Second weight codewords of generalized Reed-Muller codes

Elodie Leducq

1 Introduction

In this paper, we want to characterize the second weight codewords of generalized Reed-Muller codes.

We first introduce some notations :

Let p be a prime number, n a positive integer, $q = p^n$ and $\mathbb{F}_q$ a finite field with q elements.

If m is a positive integer, we denote by B_m^q the $\mathbb{F}_q$-algebra of the functions from $\mathbb{F}_q^m$ to $\mathbb{F}_q$ and by $\mathbb{F}_q[X_1,\dots,X_m]$ the $\mathbb{F}_q$-algebra of polynomials in m variables with coefficients in $\mathbb{F}_q$.

We consider the morphism of $\mathbb{F}_q$-algebras $\varphi : \mathbb{F}_q[X_1,\dots,X_m] \to B_m^q$ which associates to $P \in \mathbb{F}_q[X_1,\dots,X_m]$ the function $f \in B_m^q$ such that

$$\forall x = (x_1,\dots,x_m) \in \mathbb{F}_q^m,\ f(x) = P(x_1,\dots,x_m).$$

The morphism φ is onto and its kernel is the ideal generated by the polynomials $X_1^q - X_1,\dots,X_m^q - X_m$. So, for each $f \in B_m^q$, there exists a unique polynomial $P \in \mathbb{F}_q[X_1,\dots,X_m]$ such that the degree of P in each variable is at most $q-1$ and $\varphi(P) = f$. We say that P is the reduced form of f and we define the degree $\deg(f)$ of f as the degree of its reduced form. The support of f is the set $\{x \in \mathbb{F}_q^m : f(x) \neq 0\}$ and we denote by $|f|$ the cardinal of its support (by identifying canonically B_m^q and $\mathbb{F}_q^{q^m}$, $|f|$ is actually the Hamming weight of f).

For $0 \leq r \leq m(q-1)$, the rth order generalized Reed-Muller code of length q^m is

$$R_q(r,m) := \{f \in B_m^q : \deg(f) \leq r\}.$$

For $1 \leq r \leq m(q-1)-2$, the automorphism group of generalized Reed-Muller codes $R_q(r,m)$ is the affine group of $\mathbb{F}_q^m$ (see [1]).

For more results on generalized Reed-Muller codes, we can see for example [6].

We are now able to give precisely some results about minimum weight codewords and second weight codewords :

We write $r = t(q-1)+s$, $0 \leq t \leq m-1$, $0 \leq s \leq q-2$.

In [9], interpreting generalized Reed-Muller codes in terms of BCH codes, it is proved that the minimal weight of the generalized Reed-Muller code $R_q(r,m)$ is $(q-s)q^{m-t-1}$.

The following theorem gives the minimum weight codewords of generalized Reed-Muller codes and is proved in [6] or [10]

Theorem 1.1 *Let $r = t(q-1)+s < m(q-1)$, $0 \leq s \leq q-2$. The minimal weight codewords of $R_q(r,m)$ are codewords of $R_q(r,m)$ whose support is the union of $(q-s)$ distinct parallel affine subspaces of codimension $t+1$ included in an affine subspace of codimension t.*

In [8], Geil proves that the second weight of generalized Reed-Muller codes $R_q((m-1)(q-1)+s,m)$, $1 \leq s \leq q-2$ is $q-s+1$ and that the second weight of generalized Reed-Muller codes $R_q(r,m)$, $2 \leq r < q$ is $(q-r+1)(q-1)q^{m-2}$. The other cases can be found in the following theorem. Rolland proves all the cases such that $s \neq 1$ in [11]. The case where $s=1$ has been proved by Bruen in [4] using methods of Erickson (see [7]):

Theorem 1.2 *For $m \geq 3$, $q \geq 3$ and $q \leq r \leq (m-1)(q-1)$ the second weight W_2 of the generalized Reed-Muller codes $R_q(r,m)$ satisfies :*

1. *if $1 \leq t \leq m-1$ and $s = 0$,*

$$W_2 = 2(q-1)q^{m-t-1};$$

2. *if $1 \leq t \leq m-2$ and $s = 1$,*
   - *(a) if $q = 3$, $W_2 = 8 \times 3^{m-t-2}$,*
   - *(b) if $q \geq 4$, $W_2 = q^{m-t}$,*

3. *if $1 \leq t \leq m-2$ and $2 \leq s \leq q-2$,*

$$W_2 = (q-s+1)(q-1)q^{m-t-2}.$$

In [5], Cherdieu and Rolland prove that the codewords of the second weight of $R_q(s,m)$, $2 \leq s \leq q-2$, which are the product of s polynomials of degree 1 are of the following form.

Theorem 1.3 *Let $m \geq 2$, $2 \leq s \leq q-2$ and $f \in R_q(s,m)$ such that $|f| = (q-s+1)(q-1)q^{m-2}$; we denote by S the support of f. Assume that f is the product of s polynomials of degree 1 then either S is the union of $q-s+1$ parallel affine hyperplanes minus their intersection with an affine hyperplane which is not parallel or S is the union of $(q-s+1)$ affine hyperplanes which meet in a common affine subspace of codimension 2 minus this intersection.*

In [12], Sboui proves that the only codewords of $R_q(s,m)$, $2 \leq s \leq \frac{q}{2}$ whose weight is $(q-s+1)(q-1)q^{m-2}$ are these codewords.

All the results proved in this paper are summarized in Section 2 and their proofs are in the following sections.

2 Results

In the following, except when an other affine space is specified, an hyperplane or a subspace is an affine hyperplane or an affine subspace of $\mathbb{F}_q^m$.

2.1 Case where $t = m - 1$ and $s \neq 0$

Theorem 2.1 *Let $m \geq 2$, $q \geq 5$, $1 \leq s \leq q-4$ and $f \in R_q((m-1)(q-1)+s, m)$ such that $|f| = q - s + 1$. Then the support of f is included in a line.*

Proposition 2.2 *Let $m \geq 2$. If $q \geq 3$ and $f \in R_q((m-1)(q-1) + q - 3, m)$ such that $[f| = 4$ or $f \in R_q((m-1)(q-1+q-2, m)$ such that $|f| = 3$, then the support of f is included in an affine plane.*

2.2 Case where $0 \leq t \leq m - 2$ and $2 \leq s \leq q - 2$

Theorem 2.3 *Let $q \geq 4$, $m \geq 2$, $0 \leq t \leq m-2$, $2 \leq s \leq q-2$. The second weight codewords of $R_q(t(q-1) + s, m)$ are codewords of $R_q(t(q-1) + s, m)$ whose support S is included in an affine subspace of codimension t and either S is the union of $q-s+1$ parallel affine subspaces of codimension $t+1$ minus their intersection with an affine subspace of codimension $t+1$ which is not parallel or S is the union of $(q-s+1)$ affine subspaces of codimension $t+1$ which meet in an affine subspace of codimension $t+2$ minus this intersection (see Figure 1).*

Figure 1: The possible support for a second weight codeword of $R_4(5, 3)$

2.3 Case where $s = 0$

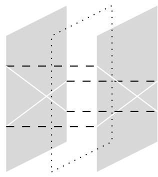
Theorem 2.4 *Let $m \geq 2$, $q \geq 3$, $1 \leq t \leq m-1$. The second weight codewords of $R_q(t(q-1), m)$ are codewords of $R_q(t(q-1), m)$ whose support S is included in an affine subspace of codimension $t-1$ and either S is the union of 2 parallel affine subspaces of codimension t minus their intersection with an affine subspace of codimension t which is not parallel or S is the union of 2 non parallel affine subspaces of codimension t minus their intersection.*

2.4 Case where $0 \leq t \leq m-2$ and $s = 1$

Theorem 2.5 *For $q \geq 4$, $m \geq 1$, $0 \leq t \leq m-1$, if $f \in R_q(t(q-1)+1, m)$ is such that $|f| = q^{m-t}$, the support of f is an affine subspace of codimension t.*

Proposition 2.6 *Let $m \geq 3$, $1 \leq t \leq m-2$ and $f \in R_3(2t+1, m)$ such that $|f| = 8.3^{m-t-2}$. We denote by S the support of f. Then S is included in A an affine subspace of dimension $m-t+1$, S is the union of two parallel hyperplanes of A minus their intersection with two non parallel hyperplanes of A (see Figure 2).*

Figure 2: The support of a second weight codeword of $R_3(3,3)$



3 Some tools

The following lemma and its corollary are proved in [6].

Lemma 3.1 *Let $m \geq 1$, $q \geq 2$, $f \in B_m^q$ and $a \in \mathbb{F}_q$. If for all $(x_2, \ldots, x_m)$ in $\mathbb{F}_q^{m-1}$, $f(a, x_2, \ldots, x_m) = 0$ then for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,*

$$f(x_1, \ldots, x_m) = (x_1 - a)g(x_1, \ldots, x_m)$$

with $\deg_{x_1}(g) \leq \deg_{x_1}(f) - 1$.

Corollary 3.2 *Let $m \geq 1$, $q \geq 2$, $f \in B_m^q$ and $a \in \mathbb{F}_q$. If for all $(x_1, \ldots, x_m)$ in $\mathbb{F}_q^m$ such that $x_1 \neq a$, $f(x_1, \ldots, x_m) = 0$ then for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$, $f(x_1, \ldots, x_m) = (1 - (x_1 - a)^{q-1})g(x_2, \ldots, x_m)$.*

Lemma 3.3 *Let $q \geq 3$, $m \geq 3$, and S be a set of points of $\mathbb{F}_q^m$ such that $\#S = u.q^n < q^m$, with $u \not\equiv 0 \mod q$. Assume that for all hyperplanes H either $\#(S \cap H) = 0$ or $\#(S \cap H) = v.q^{n-1}$, $v < u$ or $\#(S \cap H) \geq u.q^{n-1}$ Then there exists H an affine hyperplane such that S does not meet H or such that $\#(S \cap H) = vq^{n-1}$.*

Proof : Assume that for all H hyperplane, $S \cap H \neq \emptyset$ and $\#(S \cap H) \neq vq^{n-1}$. Consider an affine hyperplane H; then for all H' hyperplane parallel to H, $\#(S \cap H') \geq u.q^{n-1}$. Since $u.q^n = \#S = \sum_{H'//H} \#(S \cap H')$, we get that for all H hyperplane, $\#(S \cap H) = u.q^{n-1}$.
Now consider A an affine subspace of codimension 2 and the $(q+1)$ hyperplanes

through A. These hyperplanes intersect only in A and their union is equal to $\mathbb{F}_q^m$. So

$$uq^n = \#S = (q+1)u.q^{n-1} - q\#(S \cap A).$$

Finally we get a contradiction if $n = 1$. Otherwise, $\#(S \cap A) = u.q^{n-2}$. Iterating this argument, we get that for all A affine subspace of codimension $k \leq n$, $\#(S \cap A) = u.q^{n-k}$.
Let A be an affine subspace of codimension $n+1$ and A' an affine subspace of codimension $n-1$ containing A. We consider the $(q+1)$ affine subspace of codimension n containing A and included in A', then

$$u.q = \#(S \cap A') = (q+1)u - q\#(S \cap A)$$

which is absurd since $\#(S \cap A)$ is an integer and $u \not\equiv 0 \mod q$. So there exists H_0 an hyperplane such that $\#(S \cap H_0) = vq^{n-1}$ or S does not meet H_0.

□

4 Case where $t = m - 1$ and $s \neq 0$

4.1 Proof of Theorem 2.1

Let ω_1, $\omega_2 \in S$ and H an affine hyperplane containing ω_1 and ω_2. Assume $S \cap H \neq S$. We have $\#S = q - s + 1 \leq q$ and $\omega_1, \omega_2 \in S \cap H$, so there exists an affine hyperplane parallel to H which does not meet S. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H and we denote by H_a the affine hyperplane parallel to H of equation $x_1 = a$, $a \in \mathbb{F}_q$. Let $I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset\}$ and denote by $k := \#I$; $s \leq k \leq q - 2$. Let $c \notin I$, we define

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m,\ f_c(x) = f(x) \prod_{a \notin I, a \neq c} (x_1 - a)$$

that is to say f_c is a function in B_m^q such that its support is $S \cap H_c$. Since $c \notin I$, f_c is not identically zero. Then $|f| = \sum_{c \notin I} |f_c|$ and we consider two cases.

- Assume $k > s$.
  Then the reduced form of f_c has degree at most $(m-1)(q-1) + q - 1 + s - k$ and $|f_c| \geq k - s + 1$. Then,

  $$(q - s + 1) = |f| = \sum_{c \notin I} |f_c| \geq (q - k)(k - s + 1)$$

  which gives

  $$1 \geq (q - 1 - k)(k - s)$$

  this is possible if and only if $k = q - 2 = s + 1$ and we get a contradiction since $s \leq q - 4$.

- Assume that $k = s$.
  Then S meets $(q - s - 1)$ affine hyperplanes parallel to H in 1 point and H in 2 points. Consider the function g in B_m^q defined by

  $$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m,\ g(x) = x_1 f(x).$$

The reduced form of g has degree at most $(m-1)(q-1)+s+1$ and

$$|g| = (q-s-1).$$

So g is a minimum weight codeword of $R_q((m-1)(q-1)+s+1, m)$ and its support is included in a line. This line is not included in H. So consider H_1 an affine hyperplane which contains this line but does not contain both ω_1 and ω_2. Then $S \cap H_1 \neq S$ and H_1 contains at least 3 points of S since $s \leq q-4$ which gives a contradiction by applying the previous argument to H_1.

So S is included in all affine hyperplanes through ω_1 and ω_2 which gives the result.

4.2 Proof of Theorem 2.2

- If $f \in R_q((m-1)(q-1)+q-2, m)$ is such that $|f| = 3$, we have the result since 3 points are always included in an affine plane.
- Assume $f \in R_q((m-1)(q-1)+q-3, m)$ is such that $|f| = 4$.
  Let a, b, c, $d \in \mathbb{F}_q^*$ and $\omega^{(a)} = (\omega_1^{(a)}, \dots, \omega_m^{(a)})$, $\omega^{(b)} = (\omega_1^{(b)}, \dots, \omega_m^{(b)})$, $\omega^{(c)} = (\omega_1^{(c)}, \dots, \omega_m^{(c)})$, $\omega^{(d)} = (\omega_1^{(d)}, \dots, \omega_m^{(d)})$ 4 distinct points of $\mathbb{F}_q^m$ such that $\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m$,

$$\begin{aligned} f(x) =& a\prod_{i=1}^{m}\left(1-(x_i-\omega_i^{(a)})^{q-1}\right) + b\prod_{i=1}^{m}\left(1-(x_i-\omega_i^{(b)})^{q-1}\right) \\ &+ c\prod_{i=1}^{m}\left(1-(x_i-\omega_i^{(c)})^{q-1}\right) + d\prod_{i=1}^{m}\left(1-(x_i-\omega_i^{(d)})^{q-1}\right). \end{aligned}$$

  So,

$$\begin{aligned} f(x) =& (-1)^m(a+b+c+d)\prod_{i=1}^{m} x_i^{q-1} \\ &+ (-1)^{m-1}\sum_{i=1}^{m}(a\omega_i^{(a)} + b\omega_i^{(b)} + c\omega_i^{(c)} + d\omega_i^{(d)})x_i^{q-2}\prod_{j\neq i} x_j^{q-1} + r \end{aligned}$$

  with $\deg(r) \leq (m-1)(q-1)+q-3$. Since $f \in R_q((m-1)(q-1)+q-3, m)$,

$$\left\{\begin{array}{l} a+b+c+d = 0 \\ a\omega^{(a)} + b\omega^{(b)} + c\omega^{(c)} + d\omega^{(d)} = 0 \end{array}\right. .$$

  So, $a\overrightarrow{\omega^{(d)}\omega^{(a)}} + b\overrightarrow{\omega^{(d)}\omega^{(b)}} + c\overrightarrow{\omega^{(d)}\omega^{(c)}} = \overrightarrow{0}$ which gives the result.

Remark 4.1 *In both cases we cannot prove that the support of f is included in a line. Indeed,*

- *Let ω_1, ω_2, ω_3 3 points of $\mathbb{F}_q^m$ not included in a line. For $q \geq 3$ we can find a, $b \in \mathbb{F}_q^*$ such that $a+b \neq 0$. Let $f = a1_{\omega_1} + b1_{\omega_2} - (a+b)1_{\omega_3}$ where for $\omega \in \mathbb{F}_q^m$, 1_ω is the function from $\mathbb{F}_q^m$ to $\mathbb{F}_q$ such that $1_\omega(\omega) = 1$ and $1_\omega(x) = 0$ for all $x \neq \omega$. Then, since $\sum_{x \in \mathbb{F}_q^m} f(x) = a+b-(a+b) = 0$, $f \in R_q((m-1)(q-1)+q-2, m)$.*

- *Let ω_1, ω_2, ω_3 3 points of $\mathbb{F}_q^m$ not included in a line and set*
$$\omega_4 = \omega_1 + \omega_2 - \omega_3.$$
Then $f = 1_{\omega_1} + 1_{\omega_2} - 1_{\omega_3} - 1_{\omega_4} \in R_q((m-1)(q-1) + q - 3, m)$.

5 Case where $0 \leq t \leq m-2$ and $2 \leq s \leq q-2$

5.1 Case where $t = 0$

In this subsection, we write $r = a(q-1)+b$ with $0 \leq a \leq m-1$ and $0 < b \leq q-1$.

Lemma 5.1 *Let $q \geq 3$, $m \geq 2$, $0 \leq a \leq m-2$, $2 \leq b \leq q-1$ and $f \in R_q(a(q-1)+b, m)$ such that $|f| = (q-b+1)(q-1)q^{m-a-2}$; we denote by S the support of f. If H is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$ then either S meets all affine hyperplanes parallel to H or S meets $q-b+1$ affine hyperplanes parallel to H in $(q-1)q^{m-a-2}$ points or S meets $q-1$ affine hyperplanes parallel to H in $(q-b+1)q^{m-a-2}$ points.*

Proof : By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H and consider the q affine hyperplanes H_w of equation $x_1 = w$, $w \in \mathbb{F}_q$, parallel to H. Let $I := \{w \in \mathbb{F}_q : S \cap H_w = \emptyset\}$ and denote by $k := \#I$. Assume that $k \geq 1$. Since $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $k \leq q-2$. For all $c \in \mathbb{F}_q$, $c \notin I$, we define
$$\forall x = (x_1, \ldots, x_n) \in \mathbb{F}_q^m,\ f_c(x) = f(x) \prod_{w \in \mathbb{F}_q, w \neq c, w \notin I} (x_1 - w).$$

- Assume $b < k$.
Then $2 \leq q-1+b-k \leq q-2$ and for all $c \notin I$, the reduced form of f_c has degree at most $a(q-1)+q-1+b-k$. So $|f_c| \geq (k-b+1)q^{m-a-1}$. Hence
$$(q-1)(q-b+1)q^{m-a-2} \geq (q-k)(k-b+1)q^{m-a-1}$$
which means that $(b-k)q(q-k-1)+b-1 \geq 0$. However $(b-k) \leq -1$ and $q-k-1 \geq 1$ so $(b-k)q(q-k-1)+b-1 < 0$ which gives a contradiction.

- Assume $b \geq k$.
Then $0 \leq b-k \leq q-2$ and for all $c \notin I$, the reduced form of f_c has degree at most $(a+1)(q-1)+b-k$. So $|f_c| \geq (q-b+k)q^{m-a-2}$. Hence
$$(q-1)(q-b+1)q^{m-a-2} \geq (q-k)(q-b+k)q^{m-a-2}$$
with equality if and only if for all $c \notin I$, $|f_c| = (q-b+k)q^{m-a-2}$. Finally, we obtain that $(k-1)(k-b+1) \geq 0$ which is possible if and only if $k = 1$ or $1 \geq b-k \geq 0$. Now, we have to show that $k = s$ is impossible to prove the lemma. If $b = q-1$, since $k \leq q-2$, we have the result. Assume that $b \leq q-2$ and $b = k$. Then, for all $c \notin I$, $f_c \in R_q((a+1)(q-1), m)$. The minimum weight of $R_q((a+1)(q-1), m)$ is q^{m-a-1} and its second weight is $2(q-1)q^{m-a-2}$. We denote by $N_1 := \#\{c \notin I : |f_c| = q^{m-a-1}\}$. Since $k = b$, $N_1 \leq q-b$. Furthermore, we have
$$(q-b+1)(q-1)q^{m-a-2} \geq N_1 q^{m-a-1} + (q-b-N_1)2(q-1)q^{m-a-2}$$

which means that $N_1 \geq \frac{(q-1)(q-b-1)}{q-2} > q-b-1$. Finally, $N_1 = q-b$ and for all $c \notin I$, $|f_c| = q^{m-a-1}$. However $(q-1)(q-b+1)q^{m-a-2} > (q-b)q^{m-a-1}$ which gives a contradiction.

$\square$

Lemma 5.2 *For $m = 2$, $q \geq 3$, $2 \leq b \leq q-1$. The second weight codewords of $R_q(b,2)$ are codewords of $R_q(b,2)$ whose support S is the union of $q-b+1$ parallel lines minus their intersection with a line which is not parallel or S is the union of $(q-b+1)$ lines which meet in a point minus this point.*

Proof : To prove this lemma, we use some results on blocking sets proved by Erickson in [7] and Bruen in [4]. All these results are recalled in the Appendix of this paper. By Theorem 1.3, which is also true for $b = q-1$ (see [7, Lemma 3.12]), it is sufficient to prove that $f \in R_q(b,2)$ such that $|f| = (q-b+1)(q-1)$ is the product of linear factors.

Let $f \in R_q(b,2)$ such that $|f| \leq (q-b+1)(q-1) = q(q-b)+b-1$. We denote by S its support. Then, S is not a blocking set of order $(q-b)$ of $\mathbb{F}_q^2$ (Theorem A.3) and f has a linear factor (Lemma A.2).

We proceed by induction on b. If $b = 2$ and $f \in R_q(b,2)$ is such that $|f| \leq (q-b+1)(q-1)$, then f has a linear factor and by Lemma 3.1 f is the product of 2 linear factors. Assume that if $f \in R_q(b-1,2)$ is such that $|f| \leq (q-b+2)(q-1)$ then f is a product of linear factors. Let $f \in R_q(b,2)$ such that $|f| \leq (q-b+1)(q-1)$; then f has a linear factor. By applying an affine transformation, we can assume that for all $(x,y) \in \mathbb{F}_q^2$, $f(x,y) = y\widetilde{f}(x,y)$ with $\deg(\widetilde{f}) \leq b-1$. So, L the line of equation $y = 0$ does not meet S the support of f. Since $(q-b+1)(q-1) > q$, S is not included in a line and by Lemma 5.1, either S meets $(q-b+1)$ lines parallel to L in $(q-1)$ points or S meets $(q-1)$ lines parallel to L in $(q-b+1)$ points.

In the first case, by Lemma 3.1, we can write for all $(x,y) \in \mathbb{F}_q^2$,

$$f(x,y) = y(y-a_1)\dots(y-a_{b-2})g(x,y)$$

where a_i, $1 \leq i \leq q-2$ are $q-2$ distinct elements of $\mathbb{F}_q^*$ and $\deg(g) \leq 1$ which gives the result.

In the second case, we denote by $a \in \mathbb{F}_q$ the coefficient of x^{s-1} in $\widetilde{f}$. Then for any $\lambda \in \mathbb{F}_q^*$, since S meets all lines parallel to L but L in $q-s+1$ points, we get for all $x \in \mathbb{F}_q$,

$$f(x,\lambda) = a\lambda(x-a_1(\lambda))\dots(x-a_{b-1}(\lambda))$$

So there exists $a_1, \dots a_{b-1} \in \mathbb{F}_q[Y]$ of degree at most $q-1$ such that for all $(x,y) \in \mathbb{F}_q^2$,

$$f(x,y) = ay(x-a_1(y))\dots(x-a_{b-1}(y)).$$

Then for all $x \in \mathbb{F}_q$,

$$\widetilde{f}_0(x) = \widetilde{f}(x,0) = a(x-a_1(0))\dots(x-a_{b-1}(0))$$

and $|\widetilde{f}_0| \leq q-1$. So,

$$|\widetilde{f}| \leq |f| + |\widetilde{f}_0| \leq (q-b+2)(q-1).$$

By recursion hypothesis, $\widetilde{f}$ is the product of linear factors which finishes the proof of Lemma 5.2.

□

Proposition 5.3 *For $m \geq 2$, $q \geq 3$, $2 \leq b \leq q-1$. The second weight codewords of $R_q(b,m)$ are codewords of $R_q(b,m)$ whose support S is the union of $q-b+1$ parallel hyperplanes minus their intersection with an affine hyperplane which is not parallel or S is the union of $(q-b+1)$ hyperplanes which meet in an affine subspace of codimension 2 minus this intersection.*

Proof : We say that we are in configuration A if S is the union of $q-b+1$ parallel hyperplanes minus their intersection with an affine hyperplane which is not parallel (see Figure 1a) and that we are in configuration B if S is the union of $(q-b+1)$ hyperplanes which meet in an affine subspace of codimension 2 minus this intersection (see Figure 1b).

We prove this proposition by induction on m. The Lemma 5.2 proves the case where $m=2$. Assume that $m \geq 3$ and that second weight codeword of $R_q(b,m-1)$, $2 \leq b \leq q-1$ are of type A or type B. Let $f \in R_q(b,m)$ such that $|f| = (q-1)(q-b+1)q^{m-2}$ and we denote by S its support.

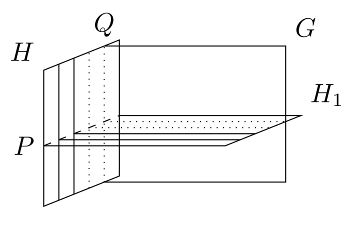
- Assume that S meets all affine hyperplanes.

  Then, by Lemma 3.3, there exists an affine hyperplane H such that $\#(S \cap H) = (q-b)q^{m-2}$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. We denote by 1_H the function in B_m^q such that

  $$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m,\ 1_H(x) = 1 - x_1^{q-1}$$

  then the reduced form $f.1_H$ has degree at most $(t+1)(q-1)+s$ and the support of $f.1_H$ is $S \cap H$ so $S \cap H$ is the support of a minimal weight codeword of $R_q(q-1+b,m)$ and $S \cap H$ is the union of $(q-b)$ parallel affine subspaces of codimension 2. Consider P an affine subspace of codimension 2 included in H such that $\#(S \cap P) = (q-b)q^{m-3}$. Assume that there are at least 2 hyperplanes through P which meet S in $(q-b)q^{m-2}$ points. Then, there exists H_1 an affine hyperplane through P different from H such that $\#(S \cap H_1) = (q-b)q^{m-2}$. So, $S \cap H_1$ is the union of $(q-b)$ parallel affine subspaces of codimension 2. Consider G an affine hyperplane which contains Q an affine subspace of codimension 2 included in H which does not meet S and the affine subspace of codimension 2 included in H_1 which meets Q but not S (see Figure 3).

Figure 3



By applying an affine transformation, we can assume that $x_m = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of an hyperplane parallel to G. For all $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B^q_{m-1}$ by

$$\forall (x_1, \dots, x_{m-1}) \in \mathbb{F}_q^{m-1}, \qquad f_\lambda(x_1, \dots, x_{m-1}) = f(x_1, \dots, x_{m-1}, \lambda).$$

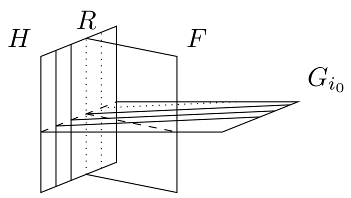
If all hyperplanes parallel to G meets S in $(q-b+1)(q-1)q^{m-3}$ then for all $\lambda \in \mathbb{F}_q$, f_λ is a second weight codeword of $R_q(b, m-1)$ and its support is of type A or B. We get a contradiction if we consider an hyperplane parallel to G which meets $S \cap H$ and $S \cap H_1$. So, there exits G_1 an hyperplane parallel to G which meets S in $(q-b)q^{m-2}$ points and $S \cap G_1$ is the union of $(q-b)$ parallel affine subspaces of codimension 2 which is a contradiction. Then for all H' hyperplane through P different from H $\#(S \cap H') \geq (q-1)(q-b+1)q^{m-3}$. Furthermore,

$$(q-b)q^{m-2} + q.(q-1)(q-b+1)q^{m-3} - q.(q-b)q^{m-3} = (q-1)(q-b+1)q^{m-2}.$$

Finally, by applying the same argument to all affine hyperplanes of codimension 2 included in H parallel to P, we get q parallel hyperplanes $(G_i)_{1 \leq i \leq q}$ such that for all $1 \leq i \leq q$, $\#(S \cap G_i) = (q-b+1)(q-1)q^{m-3}$ and $\#(S \cap G_i \cap H) = (q-b)q^{m-3}$. Then by recursion hypothesis, $S \cap G_i$ is either of type A or of type B.

If there exists i_0 such that $S \cap G_{i_0}$ is of type A. Consider F an affine hyperplane containing R an affine subspace of codimension 2 included in H which does not meet S and the affine subspace of codimension 2 included in G_{i_0} which does not meets S but meets R. If for all F' hyperplane parallel to F, $\#(S \cap F') > (q-b)q^{m-2}$ then $\#(S \cap F') = (q-1)(q-b+1)q^{m-3}$. So $S \cap F'$ is the support of a second weight codeword of $R_q(b, m-1)$ and is either of type A or of type B which is absurd is we consider an hyperplane parallel to F which meets $S \cap H$. So there exits F_1 an affine hyperplane parallel to F which meets S in $(q-b)q^{m-2}$ points. So $S \cap F_1$ is the union of $(q-s)$ parallel affine subspaces of codimension 2 which is absurd since $S \cap G_{i_0}$ is of type A (see Figure 4).

Figure 4



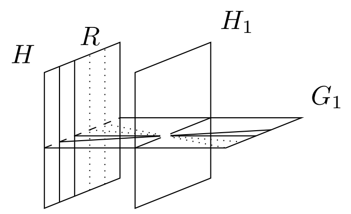
If for all $1 \leq i \leq q$, $S \cap G_i$ is of type B. Let H_1 be the affine hyperplane parallel to H which contains the affine subspace of codimension 3 intersection of the affine subspaces of codimension 2 of $S \cap G_1$. We consider R an affine subspace of codimension 2 included in H which does not meet S. Then there is $(q-b+1)$ affine hyperplanes through R which meet $S \cap G_1$

in $(q-b)q^{m-3}$. However, if we denote by k the number of hyperplanes through R which meet S in $(q-b)q^{m-2}$ points, we have

$$k(q-b)q^{m-2}+(q+1-k)(q-1)(q-b+1)q^{m-3} \leq (q-1)(q-b+1)q^{m-2}$$

which implies that $k \geq q-b+2$. For all H' hyperplane through R such that $\#(S \cap H')=(q-b)q^{m-2}$, $S \cap H'$ is the union of $(q-b)$ affine subspaces of codimension 2 parallel to R and then $\#(S \cap H' \cap G_1)=(q-b)q^{m-3}$ which is absurd (see Figure 5).

Figure 5



- So, there exists H an affine hyperplane such that H does not meet S.

Then, by Lemma 5.1, either S meets $(q-1)$ hyperplanes parallel to H in $(q-b+1)q^{m-2}$ points or S meets $(q-b+1)$ hyperplanes parallel to H in $(q-1)q^{m-2}$ points.

If S meets $(q-b+1)$ hyperplanes parallel to H in $(q-1)q^{m-2}$ points, then , for all H' hyperplane parallel to H such that $S \cap H' \neq \emptyset$, $S \cap H'$ is the support of a minimal weight codeword of $R_q(q,m)$ and is the union of $(q-1)$ parallel affine subspaces of codimension 2. Let H' be an affine hyperplane parallel to H such that $S \cap H' \neq \emptyset$. We denote by P the affine subspace of codimension 2 of H' which does not meet S. Consider H_1 an affine hyperplane which contains P and a point not in S of an affine hyperplane H" parallel to H which meets S. Then

$$\#(H_1 \setminus S) \geq bq^{m-2}+1.$$

However, if $S \cap H_1 \neq \emptyset$, $\#(H_1 \setminus S) \leq bq^{m-2}$. So, $S \cap H_1 = \emptyset$ and we are in configuration A.

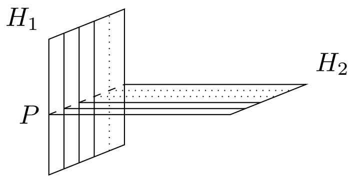
If S meets $(q-1)$ hyperplanes parallel to H in $(q-b+1)q^{m-2}$ points. Then for all H' parallel to H different from H, $S \cap H'$ is the support of a minimal weight codeword of $R_q((q-1)+b-1,m)$ and is the union of $(q-b+1)$ parallel affine subspaces of codimension 2. Let H_1 be an affine hyperplane parallel to H different from H and consider P an affine subspace of codimension 2 included in H_1 such that

$$\#(S \cap P)=(q-b+1)q^{m-3}.$$

Assume that there exists H_2 an affine hyperplane through P such that $\#(S \cap H_2)=(q-b)q^{m-2}$. Then $S \cap H_2$ is the support of a minimal

weight codeword of $R_q(q-1+b,m)$ and is the union of $(q-b)$ parallel affine subspaces of codimension 2 which is absurd since $S \cap H_2$ meets H_1 in $S \cap P$ (see Figure 6).

Figure 6



Then, for all H' through P $\#(S \cap H') \geq (q-1)(q-b+1)q^{m-3}$. Furthermore,

$$(q-b+1)q^{m-2}+q.(q-1)(q-b+1)q^{m-3}-q.(q-b+1)q^{m-3} = (q-1)(q-b+1)q^{m-2}.$$

So for all H' hyperplane through P different from H_1,

$$\#(S \cap H') = (q-1)(q-b+1)q^{m-3}.$$

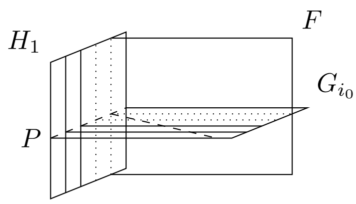
By applying the same argument to all affine subspaces of codimension 2 included in H_1 parallel to P, we get q parallel hyperplanes $(G_i)_{1 \leq i \leq q}$ such that for all $1 \leq i \leq q$, $\#(S \cap G_i) = (q-b+1)(q-1)q^{m-3}$ and $\#(S \cap G_i \cap H_1) = (q-s+1)q^{m-3}$. By recursion hypothesis, for all $1 \leq i \leq q$, either $S \cap G_i$ is of type A or $S \cap G_i$ is of type B.

Assume that there exists i_0 such that $S \cap G_{i_0}$ is of type A. Consider F an affine hyperplane containing Q an affine subspace of codimension 2 included in H_1 which does not meet S and the affine subspace of codimension 2 included in G_{i_0} which does not meets S but meets Q. Assume that S meets all hyperplanes parallel to F in at least $(q-b)q^{m-t-2}$. If for all F' parallel to F, $\#(S \cap F') > (q-b)q^{m-2}$ then

$$\#(S \cap F') \geq (q-1)(q-b+1)q^{m-3}.$$

So $S \cap F'$ is the support of a second weight codeword of $R_q(b, m-1)$ and is either of type A or of type B which is absurd is we consider an hyperplane parallel to F which meets $S \cap H_1$ and $S \cap G_{i_0}$. So, there exits F_1 an affine hyperplane parallel to F such that $\#(S \cap F_1) = (q-b)q^{m-2}$. Then, $S \cap F_1$ is the union of $(q-b)$ parallel affine subspaces of codimension 2, which is absurd. Finally, there exists an affine hyperplane parallel to F which does not meet S. By Lemma 5.1, either S meets $(q-b+1)$ hyperplanes parallel to F in $(q-1)q^{m-2}$ points and we have already seen that in this case S is of type A or S meets $(q-1)$ hyperplanes parallel to F in $(q-b+1)q^{m-2}$ points. In this case, for all F' parallel to F such that $S \cap F' \neq \emptyset$, $S \cap F'$ is the support of a minimal weight codeword of $R_q(q-1+b-1, m)$ and is the union of $q-b+1$ parallel affine subspaces of codimension 2, which is absurd since $S \cap G_{i_0}$ is of type A (see Figure 7).

Figure 7

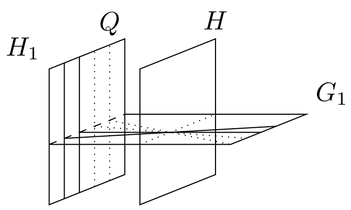


Now, assume that for all $1 \leq i \leq q$, $G_i \cap S$ is of type B. Let Q be an affine subspace of codimension 2 included in H_1 which does not meets S. Assume that S meets all affine hyperplanes through Q and denote by k the number of these hyperplanes which meet S in $(q-b)q^{m-2}$ points. Then,

$$k(q-b)q^{m-2} + (q+1-k)(q-1)(q-b+1)q^{m-3} \leq (q-1)(q-b+1)q^{m-2}$$

which means that $k \geq q-b+2$. These $(q-b+2)$ hyperplanes are minimal weight codewords of $R_q(q-1+b,m)$. So, they meet S in $(q-b)$ affine subspaces of codimension 2 parallel to Q, that is to say, they meet $S \cap G_1$ in $(q-b)q^{m-3}$ points. This is absurd since $S \cap G_1$ is of type B and so there are at most $(q-b+1)$ affine hyperplanes through Q which meet $S \cap G_1$ in $(q-b)q^{m-3}$ points (see Figure 8). So there exists an affine hyperplane through Q which does not meet S.

Figure 8



By applying the same argument to all affine subspaces of codimension 2 included in H_1 which does not meet S, since $S \cap G_i$ is of type B for all i, we get that S is of type B.

□

5.2 The support is included in an affine subspace of codimension t.

The two following lemmas are proved in [7].

Lemma 5.4 *Let $m \geq 2$, $q \geq 3$, $1 \leq t \leq m-1$, $1 \leq s \leq q-2$. Assume that $f \in R_q(t(q-1)+s,m)$ is such that $\forall x = (x_1,\dots,x_m) \in \mathbb{F}_q^m$,*

$$f(x) = (1-x_1^{q-1})\widetilde{f}(x_2,\dots,x_m)$$

and that $g \in R_q(t(q-1)+s-k)$, $1 \le k \le q-1$, is such that $(1-x_1^{q-1})$ does not divide g. Then, if $h = f+g$, either $|h| \ge (q-s+k)q^{m-t-1}$ or $k=1$.

Lemma 5.5 *Let $m \ge 2$, $q \ge 3$, $1 \le t \le m-1$, $1 \le s \le q-2$ and $f \in R_q(t(q-1)+s,m)$. For $a \in \mathbb{F}_q$, the function f_a of B^q_{m-1} defined for all $(x_2,\dots,x_m) \in \mathbb{F}_q^m$ by $f_a(x_2,\dots,x_m) = f(a,x_2,\dots,x_m)$. Assume that for a, $b \in \mathbb{F}_q$ f_a is different from the zero function and $(1-x_2^{q-1})$ divides f_a and that*

$$0 < |f_b| < (q-s+1)q^{m-t-2}.$$

Then there exists T an affine transformation, fixing x_i for $i \neq 2$ such that $(1-x_2^{q-1})$ divides $(f \circ T)_a$ and $(f \circ T)_b$.

Lemma 5.6 *Let $m \ge 3$, $q \ge 4$, $1 \le t \le m-2$ and $2 \le s \le q-2$. If $f \in R_q(t(q-1)+s,m)$ is such that $|f| = (q-s+1)(q-1)q^{m-t-2}$, then the support of f is included in an affine hyperplane of $\mathbb{F}_q^m$.*

Proof : We denote by S the support of f. Assume that S is not included in an affine hyperplane. Then, by Lemma 3.3, there exists an affine hyperplane H such that either H does not meet S or H meets S in $(q-s)q^{m-t-2}$. Now, by Lemma 5.1, since S is not included in an affine hyperplane, either S meets all affine hyperplanes parallel to H or S meets $(q-1)$ affine hyperplanes parallel to H in $(q-s+1)q^{m-t-2}$ or S meets $(q-s+1)$ affine hyperplanes parallel to H in $(q-1)q^{m-t-2}$ points. By applying an affine transformation, we can assume that $x_1 = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of H. We define $f_\lambda \in B^q_{m-1}$ by

$$\forall (x_2,\dots,x_m) \in \mathbb{F}_q^{m-1}, \qquad f_\lambda(x_2,\dots,x_m) = f(\lambda,x_2,\dots,x_m).$$

We set an order $\lambda_1,\dots,\lambda_q$ on the elements of $\mathbb{F}_q$ such that

$$|f_{\lambda_1}| \le \dots \le |f_{\lambda_q}|.$$

Then either $|f_{\lambda_1}| = 0$ or $|f_{\lambda_1}| = (q-s)q^{m-t-2}$, that is to say either f_{λ_1} is null or f_{λ_1} is the minimal weight codeword of $R_q(t(q-1)+s,m-1)$ and its support is included in an affine subspace of codimension $t+1$. Since $t \ge 1$, in both cases, the support of f_{λ_1} is included in an affine hyperplane of $\mathbb{F}_q^m$ different from the hyperplane parallel to H of equation $x_1 = \lambda_1$. By applying an affine transformation that fixes x_1, we can assume that $(1-x_2^{q-1})$ divides f_{λ_1}. Since S is not included in an affine hyperplane, there exists $2 \le k \le q$ such that $1-x_2^{q-1}$ does not divide f_{λ_k}. We denote by k_0 the smallest such k.

Assume that S meets all affine hyperplanes parallel to H and that

$$|f_{\lambda_{k_0}}| \ge (q-s+k_0-1)q^{m-t-2}.$$

Then

$$\begin{aligned}
|f| &= \sum_{k=1}^{q} |f_{\lambda_k}| \\
&\ge (q-s)q^{m-t-2}(k_0-1) + (q-k_0+1)(q-s+k_0-1)q^{m-t-2} \\
&= (q-s)q^{m-t-1} + (k_0-1)(q-k_0+1)q^{m-t-2} \\
&> (q-s)q^{m-t-1} + (s-1)q^{m-t-2}
\end{aligned}$$

which gives a contradiction. In the cases where S meets $(q-s')$, $s'=1$ or $s'=s-1$, for $1 \leq i \leq s'$, $|f_{\lambda_i}|=0$ and the support of $f_{\lambda_{s'+1}}$ is $S \cap H_{\lambda_{s'+1}}$, where $H_{\lambda_{s'+1}}$ is the hyperplane of equation $x_1=\lambda_{s'+1}$. Since $S \cap H_{\lambda_{s'+1}}$ is the support of a minimum weight codeword of $R_q((t+1)(q-1)+s',m)$, it is included in affine subspace of codimension $t+1$. So in those cases, we can assume that $k_0 \geq s'+2$. Finally, $|f_{\lambda_{k_0}}| < (q-s+k_0-1)q^{m-t-2}$.

We write

$$\begin{aligned} f(x_1,x_2,x_3,\ldots,x_m) &= \sum_{i=0}^{q-1} x_2^i g_i(x_1,x_3,\ldots,x_m) \\ &= h(x_1,x_2,x_3,\ldots,x_m)+(1-x_2^{q-1})g(x_1,x_3,\ldots,x_m). \end{aligned}$$

Since for all $1 \leq i \leq k_0-1$, $1-x_2^{q-1}$ divides f_{λ_i}, for all $(x_2,\ldots,x_m) \in \mathbb{F}_q^{m-1}$, for all $1 \leq i \leq k_0-1$, $h(\lambda_i,x_2,\ldots,x_m)=0$. So, by Lemma 3.1,

$$\begin{aligned} f(x_1,x_2,x_3,\ldots,x_m) = (x_1-\lambda_1)\ldots(x_1-\lambda_{k_0-1})\widetilde{h}(x_1,x_2,x_3,\ldots,x_m) \\ +(1-x_2^{q-1})g(x_1,x_3,\ldots,x_m) \end{aligned}$$

with $\deg(\widetilde{h}) \leq r-k_0+1$. Then by applying Lemma 5.4 to $f_{\lambda_{k_0}}$, since

$$|f_{\lambda_{k_0}}| < (q-s+k_0-1)q^{m-t-2},$$

$k_0=2$. This gives a contradiction in the cases where S does not meet all hyperplanes parallel to H. In the case where S meets all hyperplanes parallel to H, by applying Lemma 5.5, there exists T an affine transformation which fixes x_1 such that $(1-x_2^{q-1})$ divides $(f \circ T)_{\lambda_1}$ and $(f \circ T)_{\lambda_2}$, we set k_0' the smallest k such that $(1-x_2^{q-1})$ does not divide $(f \circ T)_{\lambda_k}$. Then $k_0' \geq 3$ and by applying the previous argument to $f \circ T$, we get a contradiction.

□

Proposition 5.7 *Let $m \geq 3$, $q \geq 4$, $1 \leq t \leq m-2$ and $2 \leq s \leq q-2$. If $f \in R_q(t(q-1)+s,m)$ is such that $|f|=(q-1)(q-s+1)q^{m-t-2}$, then the support of f is included in an affine subspace of codimension t.*

Proof : We denote by S the support of f. By Lemma 5.6, S is included in H an affine hyperplane. By applying an affine transformation, we can assume that $x_1=0$ is an equation of H. Let $g \in B_{m-1}^q$ defined by

$$\forall x=(x_2,\ldots,x_m) \in \mathbb{F}_q^{m-1},\ g(x)=f(0,x_2,\ldots,x_m)$$

and denote by $P \in \mathbb{F}_q[X_2,\ldots,X_m]$ its reduced form. Since

$$\forall x=(x_1,\ldots,x_m) \in \mathbb{F}_q^m,\ f(x)=(1-x_1^{q-1})P(x_2,\ldots,x_m),$$

the reduced form of $f \in R_q(t(q-1)+s,m)$ is

$$(1-X_1^{q-1})P(X_2,\ldots,X_m).$$

Then $g \in R_q((t-1)(q-1)+s,m-1)$ and

$$|g|=|f|=(q-s+1)(q-1)q^{m-t-2}=(q-1)(q-s+1)q^{m-1-(t-1)-2}.$$

Then, by Lemma 5.6, if $t \geq 2$, the support of g is included in an affine hyperplane of $\mathbb{F}_q^{m-1}$. By iterating this argument, we get that S is included in an affine subspace of codimension t.

□

5.3 Proof of Theorem 2.3

Let $0 \leq t \leq m-2$, $2 \leq s \leq q-2$ and $f \in R_q(t(q-1)+s, m)$ such that

$$|f| = (q-s+1)(q-1)q^{m-t-2};$$

we denote by S the support of f. Assume that $t \geq 1$. By Proposition 5.7, S is included in an affine subspace G of codimension t. By applying an affine transformation, we can assume that

$$G = \{x = (x_1, \dots, x_m) \in \mathbb{F}_q^m : x_i = 0 \text{ for } 1 \leq i \leq t\}.$$

Let $g \in B_{m-t}^q$ defined for all $x = (x_{t+1}, \dots, x_m) \in \mathbb{F}_q^{m-t}$ by

$$g(x) = f(0, \dots, 0, x_{t+1}, \dots, x_m)$$

and denote by $P \in \mathbb{F}_q[X_{t+1}, \dots, X_m]$ its reduced form. Since

$$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m,\ f(x) = (1 - x_1^{q-1}) \dots (1 - x_t^{q-1}) P(x_{t+1}, \dots, x_m),$$

the reduced form of $f \in R_q(t(q-1)+s, m)$ is

$$(1 - X_1^{q-1}) \dots (1 - X_t^{q-1}) P(X_{t+1}, \dots, X_m).$$

Then $g \in R_q(s, m-t)$ and $|g| = |f| = (q-s+1)(q-1)q^{m-t-2}$. Thus, using the case where $t = 0$, we finish the proof of Theorem 2.3.

6 Case where $s = 0$

6.1 The support is included in an affine subspace of dimension $m - t + 1$

Proposition 6.1 *Let $q \geq 3$, $m \geq 2$ and $f \in R_q((m-1)(q-1), m)$ such that $|f| = 2(q-1)$. Then, the support of f is included in an affine plane.*

In order to prove this proposition, we need the following lemma.

Lemma 6.2 *Let $m \geq 3$, $q \geq 4$ and $f \in R_q((m-1)(q-1), m)$ such that $|f| = 2(q-1)$. If H is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq S$, $\#(S \cap H) = N$, $3 \leq N \leq q-1$ and $S \cap H$ is not included in a line then there exists H_1 an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H_1 \neq S$, $\#(S \cap H_1) \geq N+1$ and $S \cap H_1$ is not included in a line*

Proof : Since $S \cap H \neq S$, by Lemma 5.1, either S meets $(q-1)$ hyperplanes parallel to H or S meets 2 hyperplanes parallel to H or S meets all affine hyperplanes parallel to H. If S does not meet all affine hyperplanes parallel to H then $S \cap H$ is the support of a minimal weight codeword of $R_q((m-1)(q-1)+s', m)$, $s' = 1$ or $s' = q-2$. In both cases, $S \cap H$ is included in a line which is absurd. So, S meets all affine hyperplanes parallel to H.

By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Let $I := \{a \in \mathbb{F}_q : \#(\{x_1 = a\} \cap S) = 1\}$ and $k := \#I$. Since $\#S = 2(q-1)$ and $\#(S \cap H) = N$, $k \geq N$. We define

$$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m, \quad g(x) = f(x) \prod_{a \notin I} (x_1 - a).$$

Then, $\deg(g) \leq (m-1)(q-1)+q-k$ and $|g|=k$. So, g is a minimal weight codewords of $R_q((m-1)(q-1)+q-k,m)$ and its support is included in a line L which is not included in H. We denote by $\overrightarrow{u}$ a directing vector of L. Let b be the intersection point of H and L and ω_1, ω_2, ω_3 3 points of $S \cap H$ which are not included in a line. Then there exists $\overrightarrow{v}$ and $\overrightarrow{w} \in \{\overrightarrow{b\omega_1}, \overrightarrow{b\omega_2}, \overrightarrow{b\omega_3}\}$ which are linearly independent. Since L is not included in H, $\{\overrightarrow{u}, \overrightarrow{v}, \overrightarrow{w}\}$ are linearly independent. We choose H_1 an affine hyperplane such that $b \in H_1$, $b+\overrightarrow{v} \in H_1$, $L \subset H_1$ but $b+\overrightarrow{w} \notin H_1$.

□

Now we can prove the proposition

Proof : If $m=2$, we have the result. Assume $m \geq 3$. Let S be the support of f. Since $\#S = 2(q-1) > q$, S is not included in a line. Let ω_1, ω_2, ω_3 3 points of S not included in a line. Let H be an hyperplane such that ω_1, ω_2, $\omega_3 \in H$. Assume that $S \cap H \neq S$. Then there exists an affine hyperplane H_1 such that $\#(S \cap H_1) \geq q$, $S \cap H_1$ is not included in a line and $S \cap H_1 \neq S$. Indeed, if $q=3$, we take $H_1 = H$ and for $q \geq 4$, we proceed by induction using the previous Lemma. Then by Lemma 5.1 either S meets 2 hyperplanes parallel to H_1 in 2 points or S meets 2 hyperplanes parallel to H_1 in $q-1$ points or S meets all affine hyperplanes parallel to H_1. Since $\#(S \cap H_1) \geq q$, S meets all hyperplanes parallel to H_1. Then, we must have

$$q+q-1 \leq 2(q-1)$$

which is absurd.

□

The two following lemmas are proved in [7].

Lemma 6.3 *Let $m \geq 2$, $q \geq 3$, $1 \leq t \leq m$ and $f \in R_q(t(q-1),m)$ such that $|f| = q^{m-t}$ and $g \in R_q((t(q-1)-k,m)$, $1 \leq k \leq q-1$, such that $g \neq 0$. If $h = f+g$ then either $|h| = kq^{m-t}$ or $|h| \geq (k+1)q^{m-t}$.*

Lemma 6.4 *Let $m \geq 2$, $q \geq 3$, $1 \leq t \leq m-1$ and $f \in R_q(t(q-1),m)$. For $a \in \mathbb{F}_q$, we define the function f_a of B^q_{m-1} by for all $(x_2,\ldots,x_m) \in \mathbb{F}_q^m$, $f_a(x_2,\ldots,x_m) = f(a,x_2,\ldots,x_m)$. If for some a, $b \in \mathbb{F}_q$, $|f_a| = |f_b| = q^{m-t-1}$, then there exists T an affine transformation fixing x_1 such that*

$$(f \circ T)_a = (f \circ T)_b.$$

Proposition 6.5 *Let $q \geq 3$, $m \geq 2$, $1 \leq t \leq m-1$. If $f \in R_q(t(q-1),m)$ is such that $|f| = 2(q-1)q^{m-t-1}$ then the support of f is included in an affine subspace of dimension $m-t+1$.*

Proof : For $t=1$, this is obvious. For the other cases we proceed by recursion on t. Proposition 6.1 gives the case where $t=m-1$.

If $m \leq 3$ we have considered all cases. Assume $m \geq 4$. Let $2 \leq t \leq m-2$. Assume that for $f \in R_q((t+1)(q-1),m)$ such that $|f| = 2(q-1)q^{m-t-2}$ the support of f is included in an affine subspace of dimension $m-t$. Let $f \in R_q(t(q-1),m)$ such that $|f| = 2(q-1)q^{m-t-1}$. We denote by S the support of f.

Assume that S is not included in an affine subspace of dimension $m-t+1$. Then there exists H an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq S$ and $S \cap H$ is not included in an affine space of dimension $m-t$. By Lemma 5.1, either S meets all affine hyperplanes parallel to H or S meets $(q-1)$ affine hyperplanes parallel to H in $2q^{m-t-1}$ or S meets 2 affine hyperplanes parallel to H in $(q-1)q^{m-t-1}$ points.

If S does not meet all hyperplanes parallel to H then $S \cap H$ is the support of a minimal weight codeword of $R_q(t(q-1)+s', m)$, $s' = 1$ or $s' = q-2$. So $S \cap H$ is included in an affine subspace of dimension $m-t$ which gives a contradiction.

So, S meets all affine hyperplanes parallel to H in at least q^{m-t-1} points. If for all H' parallel to H, $\#(S \cap H') > q^{m-t-1}$ then for all H' parallel to H, $\#(S \cap H') \geq 2(q-1)q^{m-t-2}$. So, for reason of cardinality, $S \cap H$ is the support of a second weight codeword of $R_q((t+1)(q-1), m)$ and by recursion hypothesis $S \cap H$ is included in an affine subspace of dimension $m-t$ which gives a contradiction. So, there exists H_1 an affine hyperplane parallel to H such that $\#(S \cap H_1) = q^{m-t-1}$.

By applying an affine transformation, we can assume that $x_1 = \lambda$, $\lambda \in \mathbb{F}_q$ is an equation of H. For $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B_{m-1}^q$ by

$$\forall (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}, \qquad f_\lambda(x_2, \ldots, x_m) = f(\lambda, x_2, \ldots, x_m).$$

We set an order $\lambda_1, \ldots, \lambda_q$ on the elements of $\mathbb{F}_q$ such that

$$|f_{\lambda_1}| \leq \ldots \leq |f_{\lambda_q}|.$$

Since $\#(S \cap H_1) = q^{m-t-1}$ and S meets all hyperplanes parallel to H,

$$|f_{\lambda_1}| = q^{m-t-1}$$

and f_{λ_1} is a minimum weight codeword of $R_q(t(q-1), m-1)$. Let k_0 be the smallest integer such that $|f_{\lambda_{k_0}}| > q^{m-t-1}$. Since $|f| > q^{m-t}$, $k_0 \leq q$. Then by Lemma 6.4 and applying an affine transformation that fixes x_1, we can assume that for all $2 \leq i \leq k_0 - 1$, $f_{\lambda_i} = f_{\lambda_1}$. If we write for all $x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x) = f_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1)\widehat{f}(x_1, \ldots, x_m).$$

Then for all $2 \leq i \leq k_0 - 1$, for all $\overline{x} = (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}$,

$$f_{\lambda_i}(\overline{x}) = f_{\lambda_1}(\overline{x}) + (\lambda_i - \lambda_1)\widehat{f}_{\lambda_i}(\overline{x}).$$

Since for all $2 \leq i \leq k_0 - 1$, $f_{\lambda_i} = f_{\lambda_1}$, by Lemma 3.1, we can write for all $x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

$$f(x) = f_{\lambda_1}(x_2, \ldots, x_m) + (x_1 - \lambda_1) \ldots (x_1 - \lambda_{k_0 - 1})\overline{f}(x_1, \ldots, x_m)$$

with $\deg(\overline{f}) \leq t(q-1) - k_0 + 1$. Now, we have $f_{\lambda_{k_0}} = f_{\lambda_1} + \lambda' \overline{f}_{\lambda_{k_0}}$, $\lambda' \in \mathbb{F}_q^*$. Then, by Lemma 6.3, either $|f_{\lambda_{k_0}}| \geq k_0 q^{m-t-1}$ or $|f_{\lambda_{k_0}}| = (k_0 - 1)q^{m-t-1}$. Assume that $|f_{\lambda_{k_0}}| \geq k_0 q^{m-t-1}$. Then

$$\begin{aligned}
|f| &= \sum_{i=1}^{q} |f_{\lambda_i}| \\
&\geq (k_0 - 1)q^{m-t-1} + (q + 1 - k_0)k_0 q^{m-t-1} \\
&= q^{m-t} + (k_0 - 1)(q - k_0 + 1)q^{m-t-1} \\
&> 2(q-1)q^{m-t-1}.
\end{aligned}$$

So, $|f_{\lambda_{k_0}}| = (k_0-1)q^{m-t-1}$. Since $|f_{\lambda_{k_0}}| > q^{m-t-1}$, $k_0 \geq 3$. Now, we have

$|f| \geq (k_0-1)q^{m-t-1} + (q+1-k_0)(k_0-1)q^{m-t-1} = (k_0-1)(q-k_0+2)q^{m-t-1}$.

So either $k_0 = q$ or $k_0 = 3$.

- Assume $k_0 = q$.
  Since $f_{\lambda_1} = \ldots = f_{\lambda_{q-1}}$ are minimum weight codeword of $R_q(t(q-1), m-1)$, there exists A an affine subspace of dimension $m-t$ of $\mathbb{F}_q^m$ such that for all $1 \leq i \leq q-1$, $S \cap H_i \subset A$, where H_i is the hyperplane parallel to H of equation $x_1 = \lambda_i$. Since S is not included in an affine subspace of dimension $m-t+1$ and $t \geq 2$, there exists an affine hyperplane G containing A such that $S \cap G \neq S$ and there exists $x \in S \cap G$, $x \notin A$. Then $\#(S \cap G) \geq (q-1)q^{m-t-1} + 1$, $S \cap G \neq S$ and $S \cap G$ is not included in an affine subspace of dimension $m-t$. Applying to G the same argument than to H, we get a contradiction.

- So, $k_0 = 3$.
  Then $f_{\lambda_1} = f_{\lambda_2}$ are minimum weight codeword of $R_q(t(q-1), m-1)$ and for reason of cardinality, for all $3 \leq i \leq q$, $|f_{\lambda_i}| = 2q^{m-t-1}$. So, there exists A an affine subspace of dimension $m-t$ of $\mathbb{F}_q^m$ such that for all $1 \leq i \leq 2$, $S \cap H_i \subset A$, where H_i is the hyperplane parallel to H of equation $x_1 = \lambda_i$. Since S is not included in an affine subspace of dimension $m-t+1$ and $t \geq 2$, there exists an affine hyperplane G containing A such that $S \cap G \neq S$ and there exists $x \in S \cap G$, $x \notin A$. Then $\#(S \cap G) \geq 2q^{m-t-1} + 1$, $S \cap G \neq S$ and $S \cap G$ is not included in an affine subspace of dimension $m-t$. Applying to G the same argument than to H, we get a contradiction.

Finally, S is included in an affine subspace of dimension $m-t+1$.

□

6.2 Proof of Theorem 2.4

Let $1 \leq t \leq m-1$ and $f \in R_q(t(q-1), m)$ such that

$$|f| = 2(q-1)q^{m-t-1};$$

we denote by S the support of f. Assume that $t \geq 2$. By proposition 6.5, S is included in an affine subspace G of codimension $t-1$. By applying an affine transformation, we can assume that

$$G = \{x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_i = 0 \text{ for } 1 \leq i \leq t-1\}.$$

Let $g \in B_{m-t+1}^q$ defined for all $x = (x_t, \ldots, x_m) \in \mathbb{F}_q^{m-t+1}$ by

$$g(x) = f(0, \ldots, 0, x_t, \ldots, x_m)$$

and denote by $P \in \mathbb{F}_q[X_t, \ldots, X_m]$ its reduced form. Since

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m,\ f(x) = (1 - x_1^{q-1}) \ldots (1 - x_{t-1}^{q-1}) P(x_t, \ldots, x_m),$$

the reduced form of $f \in R_q(t(q-1)+s, m)$ is

$$(1 - X_1^{q-1}) \ldots (1 - X_{t-1}^{q-1}) P(X_t, \ldots, X_m).$$

Then $g \in R_q(q-1, m-t+1)$ and $|g| = |f| = 2(q-1)q^{m-t-1}$. Thus, using the case where $t = 1$, we finish the proof of Theorem 2.4.

7 Case where $0 \leq t \leq m-2$ and $s = 1$

7.1 Case where $q \geq 4$

Lemma 7.1 *Let $m \geq 2$, $q \geq 4$, $0 \leq t \leq m-2$ and $f \in R_q(t(q-1)+1, m)$ such that $|f| = q^{m-t}$. We denote by S the support of f. Then, if H is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$, S meets all affine hyperplanes parallel to H.*

Proof : By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Let H_a be the q affine hyperplanes parallel to H of equation $x_1 = a$, $a \in \mathbb{F}_q$. We denote by $I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset\}$. Let $k := \#I$ and assume that $k \geq 1$. Since $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $k \leq q-2$. For all $c \notin I$ we define

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_q^m, \quad g_c(x) = f(x) \prod_{a \in \mathbb{F}_q \setminus I, a \neq c} (x_1 - a).$$

Then $|f| = \sum_{c \notin I} |g_c|$.

- Assume $k \geq 2$.
  Then for all $c \notin I$, $\deg(g_c) \leq t(q-1) + q - k$ and $2 \leq q-k \leq q-2$. So, $|g_c| \geq kq^{m-t-1}$. Let $N = \#\{c \notin I : |g_c| = kq^{m-t-1}\}$. If $|g_c| > kq^{m-t-1}$, $|g_c| \geq (k+1)(q-1)q^{m-t-2}$. Hence

  $$q^{m-t} \geq Nkq^{m-t-1} + (q-k-N)(k+1)(q-1)q^{m-t-2}.$$

  Since $k \geq 2$, we get that $N \geq q-k$. Since $(q-k)kq^{m-t-1} \neq q^{m-t}$, we get a contradiction.

- Assume $k = 1$.
  Then, for all $c \notin I$, $\deg(g_c) \leq t(q-1) + 1 + q - 2 = (t+1)(q-1)$. So $|g_c| \geq q^{m-t-1}$. Let $N = \#\{c \notin I : |g_c| = q^{m-t-1}\}$. If $|g_c| > q^{m-t-1}$, $|g_c| \geq 2(q-1)q^{m-t-2}$. Since for $q \geq 4$, $2(q-1)^2 q^{m-t-2} > q^{m-t}$, $N \geq 1$. Furthermore, since $(q-1)q^{m-t-1} < q^{m-t}$, $N \leq q-2$. For $\lambda \in \mathbb{F}_q$, we define $f_\lambda \in B_{m-1}^q$ by

  $$\forall (x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}, \qquad f_\lambda(x_2, \ldots, x_m) = f(\lambda, x_2, \ldots, x_m).$$

  We set $\lambda_1, \ldots, \lambda_q$ an order on the elements of $\mathbb{F}_q$ such that for all $i \leq N$, $|f_{\lambda_i}| = q^{m-t-1}$, $|f_{\lambda_{N+1}}| = 0$ and $q^{m-t-1} < |f_{\lambda_{N+2}}| \leq \cdots \leq |f_{\lambda_q}|$.
  Since $f_{\lambda_{N+1}} = 0$, we can write for all $(x_1, \ldots, x_m) \in \mathbb{F}_q^m$,

  $$f(x_1, \ldots, x_m) = (x_1 - \lambda_{N+1}) h(x_1, \ldots, x_m)$$

  with $\deg(h) \leq t(q-1)$. Then, for all $1 \leq i \leq q$, $i \neq N+1$ and $(x_2, \ldots, x_m) \in \mathbb{F}_q^{m-1}$,

  $$f_{\lambda_i}(x_2, \ldots, x_m) = (\lambda_i - \lambda_{N+1}) h_{\lambda_i}(x_2, \ldots, x_m).$$

  So $\deg(f_{\lambda_i}) \leq t(q-1)$ and $h_{\lambda_i} = \frac{f_{\lambda_i}}{\lambda_i - \lambda_{N+1}}$.

Since $h \in R_q(t(q-1), m)$, by Lemma 6.4, there exists an affine transformation such that for all $i \leq N$, $h_{\lambda_i} = h_{\lambda_1}$. Then, for all $(x_1, \dots, x_m) \in \mathbb{F}_q^m$,

$$h(x_1, \dots, x_m) = h_{\lambda_1}(x_2, \dots, x_m) + (x_1 - \lambda_1) \dots (x_1 - \lambda_N)\widetilde{h}(x_1, \dots, x_m)$$

with $\deg(\widetilde{h}) \leq t(q-1) - N$. Hence, for all $(x_1, \dots, x_m) \in \mathbb{F}_q^m$,

$$f(x_1, \dots, x_m) = \frac{x_1 - \lambda_{N+1}}{\lambda_1 - \lambda_{N+1}} f_{\lambda_1}(x_2 \dots, x_m) + (x_1 - \lambda_1) \dots (x_1 - \lambda_{N+1})\widetilde{h}(x_1, \dots, x_m).$$

Then, for all $(x_2, \dots, x_m) \in \mathbb{F}_q^{m-1}$,

$$f_{\lambda_{N+2}}(x_2, \dots, x_m) = \lambda f_{\lambda_1}(x_2 \dots, x_m) + \lambda' \widetilde{h}_{\lambda_{n+2}}(x_2, \dots, x_m)$$

with λ, $\lambda' \in \mathbb{F}_q^*$.

Since $f_{\lambda_1} \in R_q(t(q-1), m-1)$ and $\widetilde{h}_{\lambda_{n+2}} \in R_q(t(q-1) - N, m-1)$, by Lemma 6.3, either $|f_{\lambda_{N+2}}| = Nq^{m-t-1}$ or $|f_{\lambda_{N+2}}| \geq (N+1)q^{m-t-1}$.
If $N = 1$, since $|f_{\lambda_{N+2}}| > q^{m-t-1}$, we get

$$q^{m-t-1} + (q-2)2q^{m-t-1} \leq q^{m-t}$$

which means that $q \leq 3$. So $N \geq 2$. Then,

$$Nq^{m-t-1} + (q-1-N)Nq^{m-t-1} \leq q^{m-t}.$$

Since $N(q-N) \geq 2(q-2)$, we get that $q \leq 4$. So, the only possibility is $q = 4$ and $N = q - 2 = 2$.

If $t = 0$, H_{λ_4} contains 2.4^{m-1} points which is absurd. Assume $t \geq 1$. Since $h_{\lambda_1} = h_{\lambda_2}$ and for $i \in \{1, 2\}$, $f_{\lambda_i} = (\lambda_i - \lambda_3)h_{\lambda_i}$, $S \cap H_{\lambda_1}$ and $S \cap H_{\lambda_2}$ are both included in A an affine subspace of dimension $m - t$. If $t = 1$, by applying an affine transformation which fixes x_1, we can assume that $x_2 = 0$ is an equation of A. If S is included in A, then

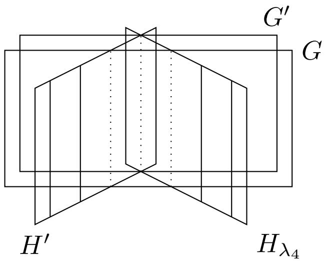
$$\#(S \cap H_{\lambda_4} \cap A) = 2.4^{m-2}$$

which is absurd since $H_{\lambda_4} \cap A$ is an affine subspace of codimension 2. So there exists an affine hyperplane H' containing A but not S. By applying an affine transformation which fixes x_1, we can assume that $x_2 = 0$ is an equation of H'. Now, consider g defined for all $(x_1, \dots, x_m) \in \mathbb{F}_q^m$ by $g(x_1, \dots, x_m) = x_2 f(x_1, \dots, x_m)$. Then $|g| \leq 2.4^{m-t-1}$. Furthermore, since S is not included in H' and $\deg(g) \leq 3t + 2$, $|g| \geq 2.4^{m-t-1}$. So g is a minimum weight codeword of $R_4(3t + 2, m)$ and its support is the union of 2 parallel affine subspace of codimension $t + 1$ included in an affine subspace of codimension t. Then, since $H' \cap H_{\lambda_4} = \emptyset$, there exists H'_1 an hyperplane parallel to H' such that $S \cap H'_1 = \emptyset$. Now, consider G the hyperplane through $H_{\lambda_4} \cap H'_1$ and $H' \cap H_{\lambda_3}$ and G' the hyperplane through $H' \cap H_{\lambda_4}$ parallel to G (see Figure 9).

Then G and G' does not meet S but S is not included in an hyperplane parallel to G which is absurd by the previous case.

□

Figure 9



Lemma 7.2 *For $m \geq 3$, if $f \in R_4(3(m-2)+1, m)$ is such that $|f| = 16$, the support of f is an affine plane.*

Proof : We denote by S the support of f.

First, we prove the case where $m = 3$. To prove this case, by Lemma 7.1, we only have to prove that there exists an affine hyperplane which does not meet S.

Assume that S meets all affine hyperplanes. Let H be an affine hyperplane. Then for all H' affine hyperplane parallel to H, $\#(S \cap H') \geq 3$. Assume that for all H' hyperplane parallel to H, $\#(S \cap H') \geq 4$. For reason of cardinality , for all H' parallel to H $\#(S \cap H') = 4$. Since $q = 4$, there exists a line in H which does not meet S. Since $3.4 + 4 = 16$, S meets 4 hyperplanes through this line in 3 points and the last one in 4 points. So, there exists H_1 an affine hyperplane such that $\#(S \cap H_1) = 3$. We denote by H_2, H_3, H_4 the hyperplanes parallel to H_1. Then, $S \cap H_1$ is the support of a minimal weight codeword of $R_4(3(m-1)+1, m)$ so $S \cap H_1$ is included in L a line. Consider L' a line in H_1 parallel to L. Then there is 4 hyperplanes through L' which meets S in 3 points and one H_1' which meets S in 4 points. Let H' be an affine hyperplane through L' which meets S in 3 points; $S \cap H'$ is minimum weight codeword of $R_4(3(m-1)+1, m)$ which does not meet H_1. So either $S \cap H'$ is included in an affine hyperplane parallel to H_1 or $S \cap H'$ meets all affine hyperplane parallel to H_1 but H_1 in 1 point. Then we consider 4 cases :

1. H_1 is the only hyperplane through L' such that $\#(S \cap H_1) = 3$ and $S \cap H_1$ is included in one of the affine hyperplane parallel to H_1.
   Since $S \cap H_1 \cap H_1' = \emptyset$ there exists an affine hyperplane parallel to H_1 which meets $S \cap H_1'$ in at least 2 points. Assume for example that it is H_2. Since $m = 3$, these 2 points are included in L_1 a line which is a translation of L. Consider H the hyperplane containing L_1 and L. Then, H meets $S \cap H_3$ and $S \cap H_4$ in 1 point (see Figure 10a). So $\#(S \cap H) = 7$

2. There are exactly 2 hyperplanes through L' which meets S in 3 points and such that its intersection with S is included in one of the affine hyperplane parallel to H_1.
   Assume that H_2 contains $S \cap \widehat{H}$ where $\widehat{H}$ is the hyperplane through L' different from H_1 such that $\#(S \cap \widehat{H}) = 3$ and $S \cap \widehat{H}$ is included in an hyperplane parallel to H_1, say H_2. We denote by $L_1 = \widehat{H} \cap H_2$. Since for

all H' hyperplane $\#(S \cap H') \geq 3$, $S \cap H'_1$ meets H_3 and H_4 in at least one point. Then consider H the hyperplane through L and L_1. Since H is different from the hyperplane through L' and L_1, H meets H_3 and H_4 in at least 1 point each (see Figure 10b). So $\#(S \cap H) \geq 7$.

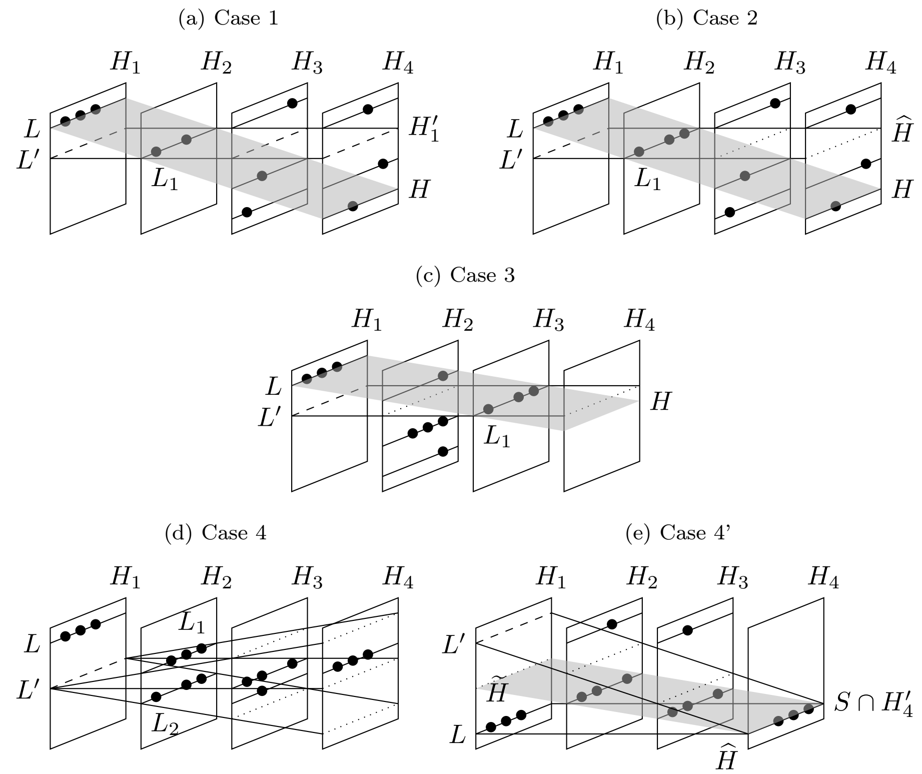
3. There are exactly 3 hyperplanes through L' which meets S in 3 points and such that its intersection with S is included in one of the affine hyperplane parallel to H_1.
   If 2 such hyperplanes have their intersection with S included in the same hyperplane parallel to H_1, say H_2, then $\#(S \cap H_2) \geq 7$. Now, assume that they are included in 2 different hyperplanes, H_2 and H_3. If $S \cap H'_1$ is included in H_4 then we consider H the hyperplane through L and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, we can assume that $S \cap H'_1$ meets H_2 in at least 1 point. Let H be the hyperplane through L and L_1 the line containing the minimum weight codeword included in H_3. Since H is different from the hyperplane through L' and L_1, H meets $S \cap H_2$ in at least 1 point and $\#(S \cap H) \geq 7$ (see Figure 10c).

4. There are 4 hyperplanes through L' which meets S in 3 points and such that its intersection with S is included in one of the affine hyperplane parallel to H_1.
   If 3 such hyperplanes have their intersection with S included in the same hyperplane parallel to H_1, say H_2, then $\#(S \cap H_2) \geq 7$. Assume that 2 such hyperplanes have their intersection included in the same hyperplane parallel to H_1, say H_2 and the last one has its intersection with S included in H_3. Then, since $\#(S \cap H_4) \geq 3$, $\#(S \cap H'_1 \cap H_4) \geq 3$.
   If $\#(S \cap H_4 \cap H'_1) = 4$, we consider H the hyperplane through L and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, there is one point of $S \cap H_4$ included in H_2 or H_3. If this point is included in H_2 then $\#(S \cap H_2) \geq 7$. If it is included in H_3, we consider L_1 and L_2 the 2 lines in H_2 containing S which are a translation of L. Then either the hyperplane through L and L_1 or the hyperplane through L and L_2 meets $S \cap H_3$ or $S \cap H_4$ (see Figure 10d). So there is an hyperplane H such that $\#(S \cap H) \geq 7$.

   Now assume that for each hyperplane H' parallel to H_1, there is only one hyperplane through L' which meets S in 3 points such that its intersection with S included in H'. If $S \cap H'_1$ is included in an affine hyperplane parallel to H_1 then we consider H the hyperplane through L and $S \cap H'_1$ and $\#(S \cap H) \geq 7$. Otherwise, $S \cap H'_1$ meets at least 2 hyperplanes parallel to H_1, say H_2 and H_3 in at least 1 point. For $i \in \{2, 3, 4\}$, we denote by H'_i the hyperplane through L' such that $S \cap H'_i \subset H_i$. If $\widehat{H}$ the hyperplane through L and $S \cap H'_4$ does not meet $S \cap H_2$ and $S \cap H_3$, then $\widetilde{H}$ the hyperplane through $S \cap H'_4$ and $S \cap H'_3$ meets $S \cap H_2$. Indeed, if $\widehat{H}$ does not meet $S \cap H_2$ we consider 4 hyperplanes through $S \cap H'_4$ different from H_4, which intersect H_2 in 4 distinct parallel lines. However 2 of these lines meet S (see Figure 10e). So there is an hyperplane H such that $\#(S \cap H) \geq 7$.

In all cases, there exists an affine hyperplane H such that $\#(S \cap H) \geq 7$. If $\#(S \cap H) > 7$, since S meets all affine hyperplanes in at least 3 points, $\#S > 7 + 3.3 = 16$ which gives a contradiction. If $\#(S \cap H) = 7$, then

Figure 10



(a) Case 1 (b) Case 2 (c) Case 3 (d) Case 4 (e) Case 4'

for all H' parallel to H different form H $\#(S \cap H') = 3$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Then $g = x_1 f \in R_4(3(m-2)+2, m)$ and $|g| = 9$. So, g is a second weight codeword of $R_4(3(m-2)+2, m)$ and by Theorem 2.3, the support of g is included in a plane P. Since S meets all hyperplanes, S is not included in P. Then, S meets all hyperplanes parallel to P in at least 3 points. However $3.3 + 9 = 18 > 16$ which is absurd.

Now, assume that $m \geq 4$. Assume that S is not included in an affine subspace of dimension 3. Then there exists H an affine hyperplane such that $S \cap H$ is not included in a plane and S is not included in H. So, by Lemma 7.1, S meets all affine hyperplanes parallel to H in at least 3 points.

Assume that for all H' parallel to H, $\#(S \cap H') \geq 4$, then for reason of cardinality, $\#(S \cap H) = 4$. So $S \cap H$ is the support of a second weight codeword of $R_4(3(m-1)+1, m)$ and is included in a plane which is absurd. So there exists H_1 an affine hyperplane parallel to H such that $\#(S \cap H_1) = 3$. Then, $S \cap H_1$ is the support of the minimum weight codeword of $R_4(3(m-1)+1, m)$ and is included in a line L. We denote by $\overrightarrow{u}$ a directing vector of L and a the point of L which is not in S.

Let w_1, w_2, w_3 3 points of $S \cap H$ which are not included in a line. Then, there are at least 2 vectors of $\{\overrightarrow{w_1w_2}, \overrightarrow{w_1w_3}, \overrightarrow{w_2w_3}\}$ which are not collinear to $\overrightarrow{u}$. Assume that they are $\overrightarrow{w_1w_2}$ and $\overrightarrow{w_1w_3}$. Let a be an affine subspace of codimension 2 included in H_1 which contains a, $a+\overrightarrow{w_1w_2}$, $a+\overrightarrow{w_1w_3}$ but not $a+\overrightarrow{u}$. Then S does not meet A. Assume that S does not meet one hyperplane through A. Then S is included in an affine hyperplane parallel to this hyperplane which is absurd by definition of A. So, S meets all hyperplanes through A and since $3.4+4 = 16$, There exists H_2 an hyperplane through A such that $\#(S \cap H_2) = 4$ and $S \cap H_2$ is included in a plane. For all H' hyperplane through A different from H_2, $\#(S \cap H') = 3$ and $S \cap H'$ is included in a line. Consider H_2' the hyperplane through A such that $w_1 \in H_2'$. Then w_1, w_2, $w_3 \in H_2'$. Since for all H' hyperplane through A different from H_2, $S \cap H'$ is included in a line and w_1, w_2, w_3 are not included in a line $H_2' = H_2$. Further more $S \cap H_2$ is included in a plane, so $S \cap H_2' \subset H$.

For all H' hyperplane through A different from H_2, $S \cap H'$ is the support of a minimum weight codeword of $R_4(3(m-1)+1, m)$ which does not meet H_1, so either $S \cap H'$ is included an affine hyperplane parallel to H_1 or $S \cap H'$ meets all affine hyperplanes parallel to H but H_1 in 1 point. Since $S \cap H_2$ is included in H and all hyperplanes parallel to H meets S in at least 3 points, there are only 3 possibilities :

1. For all H_2' hyperplane through A, $S \cap H_2'$ is included in an affine hyperplane parallel to H.
2. For H_2' hyperplane through A different from H_2 and H_1, $S \cap H_2'$ meets all affine hyperplanes parallel to H different from H_1 in 1 points.
3. There is 4 hyperplanes through A such that their intersection with S is included in an affine hyperplane parallel to H and 1 hyperplane through A which meets all hyperplanes parallel to H but H_1 in 1.

In the two first cases, since $S \cap H$ is not included in a plane and S meets all hyperplanes parallel to H in at least 3 points, $\#(S \cap H) = 7$ and for all H' parallel to H different form H, $\#(S \cap H') = 3$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Then $g = x_1 f \in R_4(3(m-2)+2, m)$ and $|g| = 9$. So, g is a second weight codeword of $R_4(3(m-2)+2, m)$ and by Theorem 2.3, the support of g is included in a plane P. Since S is not included in P, there exists H_1' an affine hyperplane which contains P but not S. Then, S meets all hyperplanes parallel to H_1' in at least 3 points. However $3.3+9 = 18 > 16$ which is absurd.

Assume we are in the third case. Since $S \cap H$ is the union of a point and $S \cap H_2$ which is included in a plane and $m \geq 4$, there exist B an affine subspace of codimension 2 included in H such that S does not meet B and $S \cap H$ is not included in affine hyperplane parallel to B. Then S meets all affine hyperplanes through B in at most 4 points which is a contradiction since $\#(S \cap H) = 5$.

So S is included in G an affine subspace of dimension 3. By applying an affine transformation, we can assume that

$$G := \{(x_1, \dots, x_m) \in \mathbb{F}_q^m : x_4 = \dots = x_m = 0\}.$$

Let $g \in B_3^q$ defined for all $x = (x_1, x_2, x_3) \in \mathbb{F}_q^3$ by

$$g(x) = f(x_1, x_2, x_3, 0, \dots, 0)$$

and denote by $P \in \mathbb{F}_q[X_1, X_2, X_3]$ its reduced form. Since

$$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m,\ f(x) = (1 - x_4^{q-1}) \dots (1 - x_m^{q-1}) P(x_1, x_2, x_3),$$

the reduced form of $f \in R_q(3(m-2)+1, m)$ is

$$(1 - X_4^{q-1}) \dots (1 - X_m^{q-1}) P(X_1, X_2, X_3).$$

Then $g \in R_q(4, 3)$ and $|g| = |f| = 16$. Thus, using the case where $m = 3$, we finish the proof of Lemma 7.2.

□

Theorem 7.3 *For $q \geq 4$, $m \geq 2$, $0 \leq t \leq m-2$, if $f \in R_q(t(q-1)+1, m)$ is such that $|f| = q^{m-t}$, the support of f is an affine subspace of codimension t.*

Proof : If $t = 0$, the second weight is q^m and we have the result.

For other cases, we proceed by recursion on t.

If $q \geq 5$, we have already proved the case where $t = m-1$ (Theorem 2.1); if $m = 2$ and $t = m-2 = 0$, we have the result. Assume that $m \geq 3$.

For $q = 4$, if $m = 2$, $t = m-2 = 0$ and we have the result. If $m \geq 3$, we have already proved the case $t = m-2$ (Lemma 7.2). Furthermore, if $m = 3$ we have considered all cases. Assume $m \geq 4$

Let $1 \leq t \leq m-2$ (or for $q = 4$, $1 \leq t \leq m-3$). Assume that the support of $f \in R_q((t+1)(q-1)+1, m)$ such that $|f| = q^{m-t-1}$ is an affine subspace of codimension $t+1$.

Let $f \in R_q(t(q-1)+1, m)$ such that $|f| = q^{m-t}$. We denote by S its support. Assume that S is not included in an affine subspace of codimension t. Then there exists H an affine hyperplane such that $S \cap H$ is not included in an affine subspace of codimension $t+1$ and $S \cap H \neq S$. Then, by Lemma 7.1, S meets all affine hyperplanes parallel to H and for all H' hyperplane parallel to H,

$$\#(S \cap H') \geq (q-1)q^{m-t-2}.$$

If for all H' hyperplane parallel to H, $\#(S \cap H') > (q-1)q^{m-t-2}$ then, for reason of cardinality, $\#(S \cap H) = q^{m-t-1}$. So $S \cap H$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$ and is included in an affine subspace of codimension $t+1$ which is a contradiction.

So there exists H_1 parallel to H such that $\#(S \cap H_1) = (q-1)q^{m-t-2}$. Then $S \cap H_1$ is the support of a minimal weight codeword of $R_q((t+1)(q-1)+1, m)$. Hence, $S \cap H_1$ is the union of $q-1$ affine subspaces of codimension $t+2$ included in an affine subspace of codimension $t+1$.

Let A be an affine subspace of codimension 2 included in H_1 such that A meets the affine subspace of codimension $t+1$ which contains $S \cap H_1$ in the affine subspace of codimension $t+2$ which does not meet S. Assume that there is an affine hyperplane through A which does not meet S. Then, by Lemma 7.1,

S is included in an affine hyperplane parallel to this hyperplane which is absurd by construction of A. So, S meets all hyperplanes through A. Furthermore,

$$q^{m-t} = q^{m-t-1} + q(q-1)q^{m-t-2}.$$

So S meets one of the hyperplane through A in q^{m-t-1} points, say H_2, and all the others in $(q-1)q^{m-t-2}$ points.
Since $H_2 \neq H_1$, $H_2 \cap H_1 = A$ and $S \cap H_2 \cap H_1 = \emptyset$. So, $S \cap H_2$ is the support of a second weight codewords of $R_q((t+1)(q-1)+1, m)$ which does not meet H_1. Hence, $S \cap H_2$ is included in one of the affine hyperplanes parallel to H. Furthermore, for all H_2' hyperplane through A different from H_2 and H_1, $S \cap H_2'$ is the support of a minimum weight codeword of $R_q((t+1)(q-1)+1, m)$ which does not meet H_1, so it meets all affine hyperplanes parallel to H_1 different from H_1 in q^{m-t-2} points or is included in an affine hyperplane parallel to H_1. Since $S \cap H_2$ is included in one of the affine hyperplanes parallel to H and all hyperplanes parallel to H meets S in at least $(q-1)q^{m-t-2}$ points, there are only 3 possibilities :

1. For all H_2' hyperplane through A, $S \cap H_2'$ is included in an affine hyperplane parallel to H.
2. For H_2' hyperplane through A different from H_2 and H_1, $S \cap H_2'$ meets all affine hyperplanes parallel to H different from H_1 in q^{m-t-2} points.
3. There is q hyperplanes through A such that their intersection with S is included in an affine hyperplane parallel to H and 1 hyperplane through A which meets all hyperplanes parallel to H but H_1 in q^{m-t-2}.

In the two first cases, if $S \cap H_2$ is not included in H' parallel to H, $\#(S \cap H') = (q-1)q^{m-t-2}$ and $S \cap H'$ is the support of a minimum weight codewords of $R_q((t+1)(q-1)+1, m)$. So $S \cap H'$ is included in an affine subspace of codimension $t+1$. Then, necessarily, $S \cap H_2$ is included in H. For all H' parallel to H but H, $\#(S \cap H') = (q-1)q^{m-t-2}$. In the third case, for all H' hyperplane parallel to H different from H_1 which does not contain $S \cap H_2$, $\#(S \cap H') = q^{m-t-1}$. So $S \cap H'$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$ and is an affine subspace of dimension $m-t-1$. Then, $S \cap H_2 \subset H$ and $\#(S \cap H) = q^{m-t-2} + q^{m-t-1}$, $\#(S \cap H_1) = (q-1)q^{m-t-2}$. So if we are in the last case for reason of cardinality, for all A' affine subspace of codimension 2 included in H_1 such that A' meets the affine subspace of codimension $t+1$ which contains $S \cap H_1$ in the affine subspace of codimension $t+2$ which does not meet S we are also in case 3. Then S is the union of affine subspaces of dimension $m-t-2$ which are a translation of the affine subspace of codimension $t+2$ which does not meet S in $S \cap H_1$. Then, since $S \cap H_2$ is the support of a second weight codeword of $R_q((t+1)(q-1)+1, m)$, it is an affine subspace of dimension $m-t-1$. So $S \cap H$ is the union of an affine subspace of dimension $m-t-1$ and an affine subspace of dimension $m-t-2$. Since S is the union of affine subspaces of dimension $m-t-2$ which are a translation of an affine subspace of codimension $t+2$, there exists B an affine subspace of codimension 2 such that B does not meet S and $S \cap H$ is not included in an affine subspace of codimension 2 parallel to B. Now, we consider all affine hyperplanes through B. Assume that there exists G an affine hyperplane through B which does not meet S. Then, S is included in an affine hyperplane parallel to G

which is absurd by construction of B. So, S meets all hyperplanes through B and there exists G_1 hyperplane through B such that $\#(S \cap G_1) = q^{m-t-1}$ and for all G through B but G_1, $\#(S \cap G) = (q-1)q^{m-t-2}$ which is absurd since $\#(S \cap H) = q^{m-t-1} + q^{m-t-2}$. Finally, we are in case 1 or 2.

By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Now, consider g the function defined by

$$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m \quad g(x) = x_1 f(x).$$

Then $\deg(g) \leq t(q-1) + 2$ and $|g| = (q-1)^2 q^{m-t-2}$. So, g is a second weight codeword of $R_q(t(q-1)+2, m)$ and by Theorem 2.3, the support of g is included in an affine subspace of codimension t.

Let H_3 be an affine hyperplane containing the support of g but not S. Then, $\#(S \cap H_3) \geq (q-1)^2 q^{m-t-2}$. Furthermore, since $S \not\subset H_3$, S meets all affine hyperplanes parallel to H_3 in at least $(q-1)q^{m-t-2}$. Finally,

$$\#S \geq 2(q-1)^2 q^{m-t-2} > q^{m-t}.$$

We get a contradiction. So S is included in an affine subspace of codimension t. For reason of cardinality, S is an affine subspace of codimension t.

□

7.2 Case where $q = 3$, proof of Theorem 2.6

Lemma 7.4 *Let $m \geq 2$, $0 \leq t \leq m-2$, $f \in R_3(2t+1, m)$ such that $|f| = 8.3^{m-t-2}$. If H is an affine hyperplane of $\mathbb{F}_q^m$ such that $S \cap H \neq \emptyset$ and $S \cap H \neq S$ then either S meets 2 hyperplanes parallel to H in 4.3^{m-t-2}points or S meets all affine hyperplanes parallel to H.*

Proof : By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. We denote by H_a the affine hyperplanes parallel to H of equation $x_1 = a$, $a \in \mathbb{F}_q$. Let $I := \{a \in \mathbb{F}_q : S \cap H_a = \emptyset\}$ and $k := \#I$. Since $S \cap H \neq \emptyset$ and $S \cap H \neq S$, $k \leq q-2 = 1$. Assume $k = 1$. For all $c \notin I$ we define

$$\forall x = (x_1, \dots, x_m) \in \mathbb{F}_q^m, \quad f_c(x) = f(x) \prod_{a \notin I, a \neq c} (x_1 - a).$$

Then $\deg(f_c) = (t+1)2$ and $|f_c| \geq 3^{m-t-1}$. Assume that there exists H' an affine hyperplane parallel to H such that $\#(S \cap H') = 3^{m-t-1}$ and $S \cap H'$ is the support of a minimal weight codeword of $R_3(2(t+1), m)$. Then consider A an affine subspace of codimension 2 included in H' containing $S \cap H'$ and A' an affine subspace of codimension 2 included in H' parallel to A. We denote by k the number of hyperplanes through A which meet S and by k' the number of affine hyperplanes through A' which meet S in 3^{m-t-1} points. Then

$$k' 3^{m-t-1} + (k-k') 4.3^{m-t-2} \leq 8.3^{m-t-2}.$$

Since $\#S > \#(S \cap H')$ and $k' \leq k$, we get $k = 2$. Then, if we denote by H'' the other hyperplane parallel to H' which meets S, $S \cap H''$ is included in an affine subspace of codimension 2 which is a translation of A. By applying this

argument to all affine subspaces of codimension 2 included in H' and containing $S \cap H'$, we get that $S \cap H''$ is included in a an affine subspace of dimension $m-t-1$. For reason of cardinality this is absurd. If $|f_c| > 3^{m-t-1}$ then $|f_c| \geq 4.3^{m-t-2}$. For reason of cardinality, we have the result.

□

Now, we prove Proposition 2.6.

- First, we prove the case where $t=1$. Obviously, S is included in an affine subspace of dimension m. Assume that S meets all affine hyperplanes of $\mathbb{F}_q^m$. Then for all H' affine hyperplane of $\mathbb{F}_q^m$, $\#(S \cap H') \geq 2.3^{m-3}$ and by Lemma 3.3, there exists H an affine hyperplane such that

$$\#(S \cap H) = 2.3^{m-3}.$$

Then $S \cap H$ is the support of a minimum weight codeword of $R_3(5,m)$. So it is the union of P_1, P_2 2 parallel affine subspaces of dimension $m-3$ included in an affine subspace of dimension $m-2$. Let A be an affine subspace of codimension 2 included in H, containing P_1 and different from the affine subspace of codimension 2 containing $S \cap H$. Then there exists A' an affine hyperplane of codimension 2 included in H parallel to A which does not meet S. We denote by k the number of affine hyperplanes through A' which meet S in 2.3^{m-3} points. Then, if $m \geq 4$,

$$k2.3^{m-3} + (4-k)8.3^{m-4} \leq 8.3^{m-3}$$

which means that $k \geq 4$. If $m=3$, $2k+(4-k)3 \leq 8$ which also means that $k \geq 4$. Then for all H' hyperplane through A different from H, $S \cap H'$ is a minimal weight codeword of $R_3(5,m)$ which does not meet H and either $S \cap H'$ is included in one of the hyperplanes parallel to H or $S \cap H'$ meets the 2 hyperplanes parallel to H different from H. In all cases, S is the union of 8 affine subspace of dimension $m-3$. By applying this argument to all affine subspaces of codimension 2 included in H, containing P_1 and different from the affine subspace of codimension 2 containing $S \cap H$, we get that these 8 affine subspaces are a translation of P_1.

Choose H_1 one of the hyperplanes through A' and consider H_2 and H_3 the 2 hyperplanes parallel to H_1. Since $\#(S \cap H_1) = 2.3^{m-3}$ and S meets all hyperplanes in at least 2.3^{m-3} points, either $\#(S \cap H_2) = 3.3^{m-3}$ and $\#(S \cap H_3) = 3.3^{m-3}$ or $\#(S \cap H_2) = 2.3^{m-3}$ and $\#(S \cap H_3) = 4.3^{m-3}$.

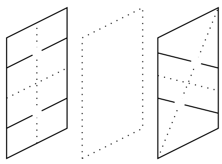
First consider the case where $\#(S \cap H_2) = 3.3^{m-3}$ and $\#(S \cap H_3) = 3.3^{m-3}$. Then there exists an affine subspace of codimension 2 in H_2 which does not meet S. We denote by k' the number of hyperplanes through A which meet S in 2.3^{m-3} points. Then , we have $k' \geq 4$ which is absurd since $\#(S \cap H_2) = 3.3^{m-3}$.

Now, consider the case where $\#(S \cap H_2) = 2.3^{m-3}$ and $\#(S \cap H_3) = 4.3^{m-3}$. By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H_3. Then $x_1.f$ is a codeword of $R_3(4,m)$ and $|x_1.f| = 4.3^{m-3}$. So, by Theorem 2.4, its support is included in an affine hyperplane H_1' and $S \cap H_1' \cap H_3 = \emptyset$. So S is included H_1' and H_3

and there exists an affine hyperplane through $H_1' \cap H_3$ which does not meet S which is absurd.

Finally there exists an affine hyperplane G_1 which does not meet S. So, by Lemma 7.4, S meets G_2 and G_3 the 2 hyperplanes parallel to G_1 in 4.3^{m-3} points. Then, Theorem 2.4, $G_2 \setminus S$ and $G_3 \setminus S$ are the union of two non parallel affine subspaces of codimension 2. Consider A one of the affine subspaces of codimension 2 in $G_2 \setminus S$. Assume that all hyperplanes through A meet S. So for all G' hyperplane through A, $\#(G' \setminus S) \leq 7.3^{m-3}$. Furthermore, one of the hyperplanes through A, say G, meets $G_3 \setminus S$ in at least 2.3^{m-3}, then $\#(G \setminus S) \geq 2.3^{m-2} + 2.3^{m-3}$ which is absurd (see Figure 11). So there exists G' through A which does not meet S. By applying the same argument to the other affine subspace of dimension 2 of $G_2 \setminus S$, we get the result for $t = 1$.

Figure 11



- We prove by recursion on t that S is included in an affine subspace of dimension $m - t + 1$. Consider first the case where $t = m - 2$. If $m = 3$ then $t = 1$ and we have already consider this case. Assume that $m \geq 4$. Let $f \in R_3(2(m-2)+1, m)$ such that $|f| = 8$. Assume that S is not included in an affine subspace of dimension 3. Let w_1, w_2, w_3, w_4 4 points of S which are not included in a plane. Since S is not included in an affine subspace of dimension 3, there exists H an affine hyperplane such that H contains w_1, w_2, w_3, w_4 but S is not included in H. Then by Lemma 7.4 either S meets 2 hyperplanes parallel to H in 4 points or S meets all hyperplanes parallel to H.

  If S meets 2 hyperplanes parallel to H then $S \cap H$ is the support of a second weight codeword of $R_3(2(m-1), m)$ so is included in a plane which is absurd since w_1, w_2, w_3, $w_4 \in S \cap H$. So S meets all hyperplanes parallel to H and for all H' hyperplane parallel to H, $\#(S \cap H') \geq 2$. Since $\#S = 8$ and $\#(S \cap H) \geq 4$, for all H' hyperplane parallel to H different from H $\#(S \cap H') = 2$ and $\#(S \cap H) = 4$.

  By applying an affine transformation, we can assume that $x_1 = 0$ is an equation of H. Then $x_1.f \in R_3(2(m-1), m)$ and $|x_1.f| = 4$ so $x_1.f$ is a second weight codeword of $R_3(2(m-1), m)$ and its support is included in a plane P not included in H. Let H' be an affine hyperplane which contains P and a point of $(S \cap H) \setminus P$ but not all the points of $S \cap H$. Then, $\#(S \cap H') \geq 5$ and $S \cap H' \neq S$. By applying the same argument to H' than to H we get a contradiction for reason of cardinality.

- If $m \leq 4$, we have already considered all the possible values for t. Assume that $m \geq 5$. Let $2 \leq t \leq m-3$. Assume that if $f \in R_3(2(t+1)+1, m)$ is such that $|f| = 8.3^{m-t-3}$ then its support is included in an affine subspace of dimension $m-t$. Let $f \in R_3(2t+1, m)$ such that $|f| = 8.3^{m-t-2}$ and denote by S its support. Assume that S is not included in an affine subspace of dimension $m-t+1$. Then, there exists H an affine hyperplane such that $S \cap H \neq S$ and $S \cap H$ is not included in an affine subspace of dimension $m-t$. So, by Lemma 7.4, either S meets 2 affine hyperplanes parallel to H in 4.3^{m-t-2} points or S meets all affine hyperplanes parallel to H.

  If S meets 2 affine hyperplanes in 4.3^{m-t-2} points, $S \cap H$ is the support of a second weight codeword of $R_3(2(t+1), m)$ and is included in an affine subspace of dimension $m-t$ which is absurd. So S meets all affine hyperplanes parallel to H and for all H' hyperplane parallel to H,

  $$\#(S \cap H') \geq 2.3^{m-t-2}.$$

  Assume that for all H' parallel to H, $\#(S \cap H') > 2.3^{m-t-2}$. Then, for reason of cardinality $\#(S \cap H) = 8.3^{m-t-3}$ and $S \cap H$ is the support of a second weight codeword of $R_3(2(t+1)+1, m)$ which is absurd since $S \cap H$ is not included in an affine subspace of dimension $m-t$. So there exists H_1 parallel to H such that $\#(S \cap H_1) = 2.3^{m-t-2}$ and $S \cap H_1$ is the support of a minimal weight codeword of $R_3(2(t+1)+1, m)$ so $S \cap H_1$ is the union of P_1 and P_2 2 parallel affine subspaces of dimension $m-t-2$ included in an affine subspace of dimension $m-t-1$.

  Let A be an affine subspace of codimension 2 included in H_1 and containing P_1 and such that $A \cap P_2 = \emptyset$. Let A' be an affine subspace of codimension 2 included in H_1 parallel to A which does not meet S. Assume that there exists H_1' an affine hyperplane through A' which does not meet S. Then, S meets H_2' and H_3' the 2 hyperplanes parallel to H_1' different from H_1' in 4.3^{m-t-2} points. For example, we can assume that $A \subset H_2'$. Then, $S \cap H_3'$ is the support of a second weight codeword of $R_3(2(t+1), m)$. So $S \cap H_3'$ meets H in 0, 3^{m-t-2}, 2.3^{m-t-2} or 4.3^{m-t-2} points. Since S meets all hyperplanes parallel to H in at least 2.3^{m-t-2} points, if

  $$\#(S \cap H \cap H_3') = 4.3^{m-t-2},$$

  $S \cap H \cap H_2' = \emptyset$. So $S \cap H$ is included in an affine subspace of dimension $m-t$ which is absurd. So $S \cap H_2'$ and $S \cap H_3'$ are the support of second weight codewords of $R_3(2(t+1), m)$ not included in H, then their intersection with H is the union of at most 2 disjoint affine subspaces of dimension $m-t-2$.

  Now assume that S meets all hyperplanes through A'. We denote by k the number of the hyperplanes through A which meet S in 2.3^{m-t-2} points. Then

  $$k2.3^{m-t-2} + (4-k)8.3^{m-t-3} \leq 8.3^{m-t-2}$$

  which means that $k \geq 4$. So for all H' affine hyperplane through A' different from H_1, $S \cap H'$ is the support of minimum weight codeword of $R_3(2(t+1)+1, m)$ which does not meet H_1. So either $S \cap H'$ is included

in H or $S \cap H'$ meets S in an affine subspace of dimension $m-t-2$. In both cases , $S \cap H$ is the union of at most 4 disjoint affine subspaces of dimension $m-t-2$. By applying this argument to all affine subspaces of dimension 2 included in H_1 containing P_1 but not P_2, we get that $S \cap H$ is the union of 4 affine subspaces of dimension $m-t-2$ which are a translation of P_1. This gives a contradiction since $S \cap H$ is not included in an affine subspace of dimension $m-t$. So S is included in an affine subspace of dimension $m-t+1$.

- Let $f \in R_3(2t+1, m)$ such that $|f| = 8.3^{m-t-2}$ and A the affine subspace of dimension $m-t+1$ containing S. By applying an affine transformation, we can assume

$$A := \{(x_1, \ldots, x_m) \in \mathbb{F}_q^m : x_1 = \ldots = x_{t-1} = 0\}.$$

Let $g \in B^3_{m-t+1}$ defined for all $x = (x_t, \ldots, x_m) \in \mathbb{F}_3^{m-t+1}$ by

$$g(x) = f(0, \ldots, 0, x_t, \ldots, x_m)$$

and denote by $P \in \mathbb{F}_3[X_t, \ldots, X_m]$ its reduced form. Since

$$\forall x = (x_1, \ldots, x_m) \in \mathbb{F}_3^m,\ f(x) = (1-x_1^2)\ldots(1-x_{t-1}^2)P(x_t, \ldots, x_m),$$

the reduced form of $f \in R_3(t(q-1)+s, m)$ is

$$(1-X_1^2)\ldots(1-X_{t-1}^2)P(X_t, \ldots, X_m).$$

Then $g \in R_3(3, m-t+1)$ and $|g| = |f| = 8.3^{m-t-2}$. Thus, using the case where $t = 1$, we finish the proof of Proposition 2.6.

A Appendix : Blocking sets

Blocking sets have been studied by Bruen in [3, 2, 4] in the case of projective planes. Erickson extends his results to affine planes in [7].

Definition A.1 *Let S be a subset of the affine space $\mathbb{F}_q^2$. We say that S is a blocking set of order n of $\mathbb{F}_q^2$ if for all line L in $\mathbb{F}_q^2$, $\#(S \cap L) \geq n$ and $\#((\mathbb{F}_q^2 \setminus S) \cap L) \geq n$.*

Proposition A.2 (Lemma 4.2 in [7]) *Let $q \geq 3$, $1 \leq b \leq q-1$ and $f \in R_q(b, 2)$. If f has no linear factor and $|f| \leq (q-b+1)(q-1)$, then the support of f is a blocking set of order $(q-b)$ of $\mathbb{F}_q^2$.*

In [7] Erickson make the following conjecture. It has been proved by Bruen in [4].

Theorem A.3 (Conjecture 4.14 in [7]) *If S is a blocking set of order n in $\mathbb{F}_q^2$, then $\#S \geq nq + q - n$.*

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